

Degree-regular triangulations of torus and Klein bottle

Basudeb Datta, Ashish Kumar Upadhyay

Department of Mathematics, Indian Institute of Science,
Bangalore 560 012, India.

dattab@math.iisc.ernet.in, upadhyay@math.iisc.ernet.in

To appear in ‘**Proc. Indian Acad. Sci. (Math. Sci.)**’

Abstract

A triangulation of a connected closed surface is called weakly regular if the action of its automorphism group on its vertices is transitive. A triangulation of a connected closed surface is called degree-regular if each of its vertices have the same degree. Clearly, a weakly regular triangulation is degree-regular. In [8], Lutz has classified all the weakly regular triangulations on at most 15 vertices. In [5], Datta and Nilakantan have classified all the degree-regular triangulations of closed surfaces on at most 11 vertices.

In this article, we have proved that any degree-regular triangulation of the torus is weakly regular. We have shown that there exists an n -vertex degree-regular triangulation of the Klein bottle if and only if n is a composite number ≥ 9 . We have constructed two distinct n -vertex weakly regular triangulations of the torus for each $n \geq 12$ and a $(4m + 2)$ -vertex weakly regular triangulation of the Klein bottle for each $m \geq 2$. For $12 \leq n \leq 15$, we have classified all the n -vertex degree-regular triangulations of the torus and the Klein bottle. There are exactly 19 such triangulations, 12 of which are triangulations of the torus and remaining 7 are triangulations of the Klein bottle. Among the last 7, only one is weakly regular.

AMS classification : 57Q15, 57M20, 57N05.

Keywords : Triangulations of 2-manifolds, regular simplicial maps, combinatorially regular triangulations, degree-regular triangulations.

1 Introduction and results.

Recall that a *simplicial complex* is a collection of non-empty finite sets (set of *vertices*) such that every non-empty subset of an element is also an element. For $i \geq 0$, the elements of size $i + 1$ are called the *i-simplices* of the simplicial complex. 1-simplices are also called the *edges* of the simplicial complex. For a simplicial complex X , the maximum of k such that X has a k -simplex is called the *dimension* of X . The set $V(X)$ of vertices of X is called the *vertex-set* of X . A simplicial complex X is called finite if $V(X)$ is finite.

If X and Y are two simplicial complexes, then a (simplicial) isomorphism from X to Y is a bijection $\varphi : V(X) \rightarrow V(Y)$ such that for $\sigma \subseteq V(X)$, σ is a simplex of X if and only if $\varphi(\sigma)$ is a simplex of Y . Two simplicial complexes X, Y are called (simplicially) *isomorphic* (and is denoted by $X \cong Y$) when such an isomorphism exists. We identify two complexes if they are isomorphic. An isomorphism from a simplicial complex X to itself is called an *automorphism* of X . All the automorphisms of X form a group, which is denoted by $\text{Aut}(X)$.

A simplicial complex X is usually thought of as a prescription for constructing a topological space (called the *geometric carrier* of X and is denoted by $|X|$) by pasting together geometric simplices. Formally, $|X|$ is the subspace of $[0, 1]^{V(X)}$ consisting of the functions $f : V(X) \rightarrow [0, 1]$ such that the support $\{v \in V(X) : f(v) \neq 0\}$ is a simplex of X and $\sum_{v \in V(X)} f(v) = 1$. If σ is a simplex then $|\sigma| := \{f \in |X| : \sum_{v \in \sigma} f(v) = 1\}$ is called the *geometric carrier* of σ . We say that a simplicial complex X *triangulates* a topological space P (or X is a *triangulation* of P) if P is homeomorphic to $|X|$. A simplicial complex X is called *connected* if $|X|$ is connected. A 2-dimensional simplicial complex is called a *combinatorial 2-manifold* if it triangulates a closed surface. A combinatorial 2-manifold X is called *orientable* if $|X|$ is an orientable 2-manifold.

If v is a vertex of a simplicial complex X , then the number of edges containing v is called the *degree* of v and is denoted by $\deg_X(v)$ (or $\deg(v)$). If the number of i -simplices of an m -dimensional finite simplicial complex X is $f_i(X)$ ($0 \leq i \leq m$), then the number $\chi(X) := \sum_{i=0}^m (-1)^i f_i(X)$ is called the *Euler characteristic* of X . A simplicial complex is called *neighbourly* if each pair of vertices form an edge.

A *combinatorially regular* combinatorial 2-manifold is a connected combinatorial 2-manifold with a flag-transitive automorphism group (a *flag* is a triple (u, e, F) , where e is an edge of the face F and u is a vertex of e). A connected combinatorial 2-manifold X is said to be *weakly regular* (or a *weakly regular triangulation* of $|X|$) if the automorphism group of X acts transitively on $V(X)$. Clearly, a combinatorially regular combinatorial 2-manifold is weakly regular. Well known examples of combinatorially regular combinatorial 2-manifolds are the boundaries of the tetrahedron, the octahedron, the icosahedron and the 6-vertex real projective plane ([4, 5]). The combinatorial manifolds $T_{3,3,0}$ and $T_{6,2,2}$ (in Examples 2 & 3) are combinatorially regular. Schulte and Wills ([10, 11]) have constructed two combinatorially regular triangulations of the orientable surface of genus 3. In [8], Lutz has shown that there are exactly 14 combinatorially regular combinatorial 2-manifolds on at most 22 vertices. By using computer, Lutz has shown the following:

PROPOSITION 1. *There are exactly 77 weakly regular combinatorial 2-manifolds on at most 15 vertices; 42 of these are orientable and 35 are non-orientable. Among these 77 combinatorial 2-manifolds, 20 are of Euler characteristic 0. These 20 are $T_{7,1,2}, \dots, T_{15,1,2}, T_{12,1,3}, \dots, T_{15,1,3}, T_{12,1,4}, T_{15,1,4}, T_{15,1,5}, T_{6,2,2}, T_{3,3,0}, Q_{5,2}$ and $Q_{7,2}$ of Examples 1, 2, 3, 6.*

A connected combinatorial 2-manifold X is said to be *degree-regular of type d* if each vertex of X has degree d . A combinatorial 2-manifold X is said to be *degree-regular* (or a *degree-regular triangulation* of $|X|$) if it is degree-regular of type d for some d . So, trivial examples of degree-regular combinatorial 2-manifolds are weakly regular and neighbourly combinatorial 2-manifolds.

If K is an n -vertex degree-regular of type d combinatorial 2-manifold then $nd = 2f_1(K) = 3f_2(K)$ and $\chi(K) = f_0(K) - f_1(K) + f_2(K) = n - \frac{nd}{2} + \frac{nd}{3} = \frac{n(6-d)}{6}$. So, if $\chi(K) \neq 0$ then only finitely many (n, d) satisfies the above equation and hence only finitely many degree-regular combinatorial 2-manifolds of a given non-zero Euler characteristic. If K is degree-regular and $\chi(K) > 0$ then $(n, d) = (4, 3), (6, 4), (6, 5)$ or $(12, 5)$. For each $(n, d) \in \{(4, 3), (6, 4), (6, 5), (12, 5)\}$, there exists unique combinatorial 2-manifold, namely, the 4-vertex 2-sphere, the boundary of the octahedron, the 6-vertex real projective plane and the boundary of the icosahedron (cf. [4, 5]). These 4 combinatorial 2-manifold are combinatorially regular. For the existence of degree-regular of type d combinatorial 2-manifolds of negative Euler characteristic, d must be at least 7. Since $\frac{n(6-d)}{6} \neq -1$ for $n > d \geq 7$, there does not exist any degree-regular combinatorial 2-manifolds of Euler characteristic

–1. If $\chi(K) = -2$ then $(f_0(K), d) = (12, 7)$. In [6], we have seen that there are exactly 6 degree-regular triangulations of the orientable surface of genus 2, three of which are weakly regular and none of them are combinatorially regular.

For the existence of an n -vertex neighbourly combinatorial 2-manifold, $n(n - 1)$ must be divisible by 6, equivalently, $n \equiv 0$ or $1 \pmod{3}$. Ringel and Jungerman ([7, 9]) have shown that *there exist neighbourly combinatorial 2-manifolds on $3k$ and $3k + 1$ vertices, for each $k \geq 2$* . By using computer, Altshuler *et al* ([3]) have shown that *there are exactly 59 orientable neighbourly combinatorial 2-manifolds on 12 vertices*. In [2], Altshuler describe two operations by which one gets many neighbourly combinatorial 2-manifolds from one such on same number of vertices. Using this he has constructed 40615 distinct non-orientable neighbourly combinatorial 2-manifolds on 12 vertices.

Here we are interested in the cases when the Euler characteristic is 0 (i.e., triangulations of the torus and the Klein bottle). Clearly, If K is an n -vertex degree-regular combinatorial 2-manifold and $\chi(K) = 0$ then $n > d = 6$. From [5], we know the following :

PROPOSITION 2. (a) *For each $n \geq 7$, there exists an n -vertex weakly regular triangulation of the torus.*

(b) *For each $k, l \geq 3$, there exists a kl -vertex degree-regular triangulation of the Klein bottle.*

PROPOSITION 3. *There are exactly 27 degree-regular combinatorial 2-manifolds on at most 11 vertices; 8 of which are of Euler characteristic 0. These 8 are $T_{7,1,2}, \dots, T_{11,1,2}, T_{3,3,0}, B_{3,3}$ and $Q_{5,2}$ of Examples 1, 3, 4, 6.*

Here we prove.

THEOREM 1. *Any degree-regular triangulation of the torus is weakly regular.*

THEOREM 2. *There exists an n -vertex degree-regular triangulation of the Klein bottle if and only if n is a composite number ≥ 9 .*

THEOREM 3. (a) *For each $n \geq 12$ there exist 2 distinct n -vertex weakly regular triangulations of the torus.*

(b) *For each $n \geq 18$ there exist 3 distinct n -vertex weakly regular triangulations of the torus.*

(c) *For each $m \geq 2$ there exists a $(4m + 2)$ -vertex weakly regular triangulation of the Klein bottle.*

THEOREM 4. *Let $T_{n,1,k}$ be as in Example 1. For a prime $n \geq 7$, if M is an n -vertex weakly regular triangulation of the torus then M is isomorphic to $T_{n,1,k}$ for some k .*

COROLLARY 5. (a) *For $n = 13$ or 17 , there are exactly 2 distinct n -vertex degree-regular combinatorial 2-manifolds of Euler characteristic 0. These are $T_{n,1,2}$ and $T_{n,1,3}$.*

(b) *There are exactly 3 distinct 19-vertex degree-regular combinatorial 2-manifolds of Euler characteristic 0. These are $T_{19,1,2}, T_{19,1,3}$ and $T_{19,1,7}$.*

From Theorem 1 and Proposition 1 we know all the degree-regular triangulations of the torus on at most 15 vertices. Here we present (without using computer) the following :

THEOREM 6. *Let M be an n -vertex degree-regular combinatorial 2-manifold of Euler characteristic 0. If $n = 12, 14$ or 15 then M is isomorphic to $T_{12,1,2}, \dots, T_{12,1,4}, T_{6,2,2}, T_{14,1,2}, T_{14,1,3}, T_{15,1,2}, \dots, T_{15,1,5}, Q_{7,2}, Q_{5,3}, B_{3,4}, B_{4,3}, B_{3,5}, B_{5,3}$ or $K_{3,4}$. These 17 combinatorial 2-manifolds are pairwise non-isomorphic. The first 10 triangulate the torus and the remaining 7 triangulate the Klein bottle. Among the last 7, only $Q_{7,2}$ is weakly regular.*

2 Examples.

In this section we present some degree-regular combinatorial 2-manifolds of Euler characteristic 0. First we give some definitions and notations which will be used throughout the paper.

A 2-simplex in a 2-dimensional simplicial complex is also said to be a *face*. We denote a face $\{u, v, w\}$ by uvw . We also denote an edge $\{u, v\}$ by uv .

A *graph* is a simplicial complex of dimension at most one. The complete graph on n vertices is denoted by K_n . Disjoint union of m copies of K_n is denoted by mK_n . A graph without any edge is called a *null graph*. An n -vertex null graph is denoted by \emptyset_n .

If G is a graph and $n \geq 0$ is an integer then we define the graph $G_n(G)$ as follows. The vertices of $G_n(G)$ are the vertices of G . Two vertices u and v form an edge in $G_n(G)$ if the number of common neighbours of u and v is n . Clearly, if G and H are isomorphic then $G_n(G)$ and $G_n(H)$ are isomorphic for all $n \geq 0$.

A connected finite graph is called a *cycle* if the degree of each vertex is 2. An n -cycle is a cycle on n vertices and is denoted by C_n (or by $C_n(a_1, \dots, a_n)$ if the edges are $a_1a_2, \dots, a_{n-1}a_n, a_na_1$). Disjoint union of m copies of C_n is denoted by mC_n .

For a simplicial complex K , the graph consisting of the edges and vertices of K is called the *edge-graph* of K and is denoted by $\text{EG}(K)$. The complement of $\text{EG}(K)$ is called the *non-edge graph* of K and is denoted by $\text{NEG}(K)$. Let K be a simplicial complex with vertex-set $V(K)$. If $U \subseteq V(K)$ then the *induced subcomplex* of K on U , denoted by $K[U]$, is the subcomplex whose simplices are those of K which are subsets of U .

If v is a vertex of a simplicial complex X , then the *link* of v in X , denoted by $\text{lk}_X(v)$ (or $\text{lk}(v)$), is the simplicial complex $\{\tau \in X : v \notin \tau, \{v\} \cup \tau \in X\}$. If v is a vertex of a simplicial complex X , then the *star* of v in X , denoted by $\text{st}_X(v)$ (or $\text{st}(v)$), is the simplicial complex $\{\{v\}, \tau, \tau \cup \{v\} : \tau \in \text{lk}_X(v)\}$. Clearly, a finite simplicial complex K is a combinatorial 2-manifold if and only if $\text{lk}_K(v)$ is a cycle for each vertex v of K .

EXAMPLE 1 : A series of weakly regular orientable combinatorial 2-manifolds of Euler characteristic 0. For each $n \geq 7$ and each $k \in \{2, \dots, \lfloor \frac{n-3}{2} \rfloor\} \cup \{\lceil \frac{n+1}{2} \rceil, \dots, n-3\}$,

$$T_{n,1,k} = \{\{i, i+k, i+k+1\}, \{i, i+1, i+k+1\} : 1 \leq i \leq n\},$$

where $V(T_{n,1,k}) = \{1, \dots, n\}$. Since $\text{lk}(i) = C_6(i+k, n+i-1, n+i-k-1, n+i-k, i+1, i+k+1)$, $T_{n,1,k}$ is a combinatorial 2-manifold. Clearly, $T_{n,1,k}$ triangulates the torus and hence it is orientable. Since \mathbb{Z}_n acts transitively (by addition) on vertices, $T_{n,1,k}$ is weakly regular. (Here addition is modulo n .) In [1], Altshuler has shown that $T_{n,1,k}$ is a subcomplex of an n -vertex cyclic polytopal 3-sphere.

LEMMA 2.1. *Let $T_{n,1,k}$ be as above. We have the following :*

- (a) $T_{n,1,k} = T_{n,1,n-k-1}$ for all n and k .
- (b) $T_{n,1,2} \not\cong T_{n,1,3}$ for all $n \geq 12$.
- (c) $T_{n,1,2} \not\cong T_{n,1,4} \not\cong T_{n,1,3}$ for all $n \geq 20$.
- (d) $T_{12,1,2} \not\cong T_{12,1,4} \not\cong T_{12,1,3}$.
- (e) $T_{13,1,4} \cong T_{13,1,2} \cong T_{13,1,5}$.
- (f) $T_{15,1,k} \not\cong T_{15,1,j}$ for $j, k \in \{2, 3, 4, 5\}$ and $j \neq k$.

- (g) $T_{16,1,5} \cong T_{16,1,2} \not\cong T_{16,1,6} \not\cong T_{16,1,3} \cong T_{16,1,4}$.
- (h) $T_{17,1,5} \cong T_{17,1,7} \cong T_{17,1,2}$ and $T_{17,1,4} \cong T_{17,1,6} \cong T_{17,1,3}$.
- (i) $T_{18,1,6} \cong T_{18,1,5}$ and $T_{18,1,k} \not\cong T_{18,1,j}$ for $j, k \in \{2, 3, 4, 5, 7\}$, $j \neq k$.
- (j) $T_{19,1,6} \cong T_{19,1,8} \cong T_{19,1,2} \not\cong T_{19,1,7} \not\cong T_{19,1,3} \cong T_{19,1,4} \cong T_{19,1,5}$.
- (k) $T_{20,1,6} \cong T_{20,1,2}$, $T_{20,1,7} \cong T_{20,1,3}$ and $T_{20,1,k} \not\cong T_{20,1,j}$ for $j, k \in \{2, 3, 4, 5, 8\}$, $j \neq k$.

Proof. Observe that $\text{lk}_{T_{n,1,k}}(i) = C_6(i+1, i+k+1, i+k, n+i-1, n+i-k-1, n+i-k) = \text{lk}_{T_{n,1,n-k-1}}(i)$. So, the faces in both $T_{n,1,k}$ and $T_{n,1,n-k-1}$ are same. This proves (a).

Then $G_4(\text{EG}(T_{n,1,2})) = C_n(1, \dots, n)$ for $n \geq 11$ and $G_4(\text{EG}(T_{n,1,3}))$ is a null graph for $n = 13$ and for $n \geq 15$. Also, $G_4(\text{EG}(T_{14,1,2}))$ is $7K_2$. So, $T_{n,1,2} \not\cong T_{n,1,3}$ for all $n \geq 13$.

Observe that $G_4(\text{EG}(T_{12,1,3}))$ is a 12-cycle with edges $\{i, i+5\}$, $1 \leq i \leq 12$. So, the edges of $G_4(\text{EG}(T_{12,1,3}))$ are non-edges of $T_{12,1,2}$. But $G_4(\text{EG}(T_{12,1,2}))$ is a subgraph of $\text{EG}(T_{12,1,2})$. So, $T_{12,1,2} \not\cong T_{12,1,3}$. This proves (b).

For all $n \geq 20$, $G_3(\text{EG}(T_{n,1,4}))$ is a null graph, but $\{i, i+2\}$ is an edge in $G_3(\text{EG}(T_{n,1,3}))$. So, $G_3(\text{EG}(T_{n,1,4})) \not\cong G_3(\text{EG}(T_{n,1,3}))$ and hence $T_{n,1,4} \not\cong T_{n,1,3}$ for $n \geq 20$.

Again, for all $n \geq 20$, $G_4(\text{EG}(T_{n,1,4})) = \emptyset_n$. So, $G_4(\text{EG}(T_{n,1,4})) \not\cong G_4(\text{EG}(T_{n,1,2}))$ and hence $T_{n,1,4} \not\cong T_{n,1,2}$ for $n \geq 20$. This proves (c).

Observe that $G_4(\text{EG}(T_{12,1,4})) = 3K_4$ (with edges $\{i, j\}$, $i - j \equiv 0 \pmod{3}$). Since, $G_4(\text{EG}(T_{12,1,2}))$ and $G_4(\text{EG}(T_{12,1,3}))$ are 12-cycles, $T_{12,1,2} \not\cong T_{12,1,4} \not\cong T_{12,1,3}$. This proves (d).

The map $i \mapsto 4i \pmod{13}$ defies an isomorphism from $T_{13,1,2}$ to $T_{13,1,4}$ and the map $i \mapsto 7i \pmod{13}$ defies an isomorphism from $T_{13,1,2}$ to $T_{13,1,5}$. This proves (e).

Observe that $G_0(\text{EG}(T_{15,1,2}))$ is a 15-cycle (with edges $\{i, i+7\}$, $1 \leq i \leq 15$) and $G_0(\text{EG}(T_{15,1,i}))$ is a null graph for $i = 3, 4, 5$. So, $T_{15,1,i} \not\cong T_{15,1,2}$ for $i = 3, 4$ or 5 . Again, $G_4(\text{EG}(T_{15,1,3}))$ is a null graph, $G_4(\text{EG}(T_{15,1,4})) = 3C_5$ (with edges $\{i, i+6\}$, $1 \leq i \leq 15$) and $G_4(\text{EG}(T_{15,1,5})) = C_{15}$ (with edges $\{i, i+4\}$, $1 \leq i \leq 15$). So, $T_{15,1,3} \not\cong T_{15,1,4} \not\cong T_{15,1,5} \not\cong T_{15,1,3}$. These prove (f).

The map $i \mapsto 3i \pmod{16}$ defines an isomorphism from $T_{16,1,4}$ to $T_{16,1,3}$ and an isomorphism from $T_{16,1,5}$ to $T_{16,1,2}$.

Observe that $G_4(\text{EG}(T_{16,1,6})) = 8K_2$ (with edges $\{i, i+8\}$, $1 \leq i \leq 8$). Since, $G_4(\text{EG}(T_{16,1,2})) = C_{16}$ and $G_4(\text{EG}(T_{16,1,3}))$ is a null graph, $T_{16,1,2} \not\cong T_{16,1,6} \not\cong T_{16,1,3}$. This proves (g).

The map $i \mapsto 14i \pmod{17}$ defines an isomorphism from $T_{17,1,5}$ to $T_{17,1,2}$. The map $i \mapsto 2i \pmod{17}$ defines an isomorphism from $T_{17,1,7}$ to $T_{17,1,2}$. The map $i \mapsto 13i \pmod{17}$ defines an isomorphism from $T_{17,1,4}$ to $T_{17,1,3}$. The map $i \mapsto 3i \pmod{17}$ defines an isomorphism from $T_{17,1,6}$ to $T_{17,1,3}$. This proves (h).

The map $i \mapsto 5i \pmod{18}$ defines an isomorphism from $T_{18,1,6}$ to $T_{18,1,5}$.

Now, $G_3(\text{EG}(T_{18,1,3})) = 2C_9$ (with edges $\{i, i+2\}$, $1 \leq i \leq 18$) and $G_3(\text{EG}(T_{18,1,7})) = 9K_2$ (with edges $\{i, i+9\}$, $1 \leq i \leq 9$). So, $T_{18,1,3} \not\cong T_{18,1,7}$. Again, $G_4(\text{EG}(T_{18,1,2})) = C_{18}(1, 2, \dots, 18) \subseteq \text{EG}(T_{18,1,2})$, $G_4(\text{EG}(T_{18,1,3}))$ and $G_4(\text{EG}(T_{18,1,7}))$ are null graphs, $G_4(\text{EG}(T_{18,1,4})) = 9K_2$ (with edges $\{i, i+9\}$, $1 \leq i \leq 9$) and $G_4(\text{EG}(T_{18,1,5})) = C_{18}$ (with edges $\{i, i+7\}$, $1 \leq i \leq 18$). Thus, $G_4(\text{EG}(T_{18,1,5}))$ is not a subgraph of $\text{EG}(T_{18,1,5})$. These imply (i).

The map $i \mapsto 3i \pmod{19}$ defines an isomorphism from $T_{19,1,5}$ to $T_{19,1,3}$. The map $i \mapsto 15i \pmod{19}$ defines an isomorphism from $T_{19,1,4}$ to $T_{19,1,3}$. The map $i \mapsto 6i \pmod{19}$ defines an isomorphism from $T_{19,1,6}$ to $T_{19,1,3}$.

19) defines an isomorphism from $T_{19,1,2}$ to $T_{19,1,6}$. The map $i \mapsto 9i \pmod{19}$ defines an isomorphism from $T_{19,1,2}$ to $T_{19,1,8}$.

The graph $G_4(\text{EG}(T_{19,1,7}))$ is null, where as $G_4(\text{EG}(T_{19,1,2})) = C_{19}$. So, $T_{19,1,7} \not\cong T_{19,1,2}$. Again, $G_0(\text{EG}(T_{19,1,7}))$ is null, where as $G_0(\text{EG}(T_{19,1,3})) = C_{19}$ (with edges $\{i, i+9\}$, $1 \leq i \leq 19$). So, $T_{19,1,7} \not\cong T_{19,1,3}$. This proves (j).

The map $i \mapsto 3i \pmod{20}$ defines an isomorphism from $T_{20,1,6}$ to $T_{20,1,2}$ and an isomorphism from $T_{20,1,7}$ to $T_{20,1,3}$.

Observe that $G_4(\text{EG}(T_{20,1,2})) = C_{20}$, $G_4(\text{EG}(T_{20,1,3}))$, $G_4(\text{EG}(T_{20,1,4}))$, $G_4(\text{EG}(T_{20,1,5}))$ are null graphs and $G_4(\text{EG}(T_{20,1,8})) = 10K_2$ (with edges $\{i, i+10\}$, $1 \leq i \leq 10$). So, $T_{20,1,2} \not\cong T_{20,1,i}$ for $i = 3, 4, 5, 8$ and $T_{20,1,8} \not\cong T_{20,1,i}$ for $i = 3, 4, 5$.

Again, $G_3(\text{EG}(T_{20,1,3})) = 2C_{10}$ (with edges $\{i, i+2\}$, $1 \leq i \leq 20$) but $G_3(\text{EG}(T_{20,1,4}))$ and $G_3(\text{EG}(T_{20,1,5}))$ are null graphs. So, $T_{20,1,3} \not\cong T_{20,1,i}$ for $i = 4, 5$.

Finally, if possible let φ be an isomorphism from $T_{20,1,4}$ to $T_{20,1,5}$. Then φ induces isomorphism between $G_n(\text{EG}(T_{20,1,4}))$ and $G_n(\text{EG}(T_{20,1,5}))$ for each n . Since, $\text{Aut}(T_{20,1,4})$ acts transitively on $V(T_{20,1,4})$, we can assume that $\varphi(20) = 20$. Since $G_0(\text{EG}(T_{20,1,4})) = C_{20}(20, 7, 14, \dots, 13)$ and $G_0(\text{EG}(T_{20,1,5})) = C_{20}(20, 3, 6, \dots, 17)$, $\varphi(7) = 17$ or 3. If $\varphi(7) = 17$ then $\varphi(14) = 14$, $\varphi(1) = 11, \dots$. In that case, $\varphi(\{20, 1\}) = \{20, 11\}$. This is a contradiction since $\{20, 1\}$ is an edge in $T_{20,1,4}$ but $\{20, 11\}$ is not an edge in $T_{20,1,5}$. Similarly, we get a contradiction if $\varphi(7) = 3$. This proves (k). \square

Let D_n denote the dihedral group of order $2n$ and $\mathbb{Z}_{m^2-m+1}:\mathbb{Z}_6 := <\rho, \mu : \rho^{m^2-m+1} = 1 = \mu^6, \mu^{-1}\rho\mu = \rho^m>$ for $m \geq 3$. In [8], Lutz has shown that $\text{Aut}(T_{n,1,k}) = D_n$ for $(n, k) = (9, 2), \dots, (15, 2), (12, 3), (14, 3), (15, 5)$, $\text{Aut}(T_{12,1,4}) = D_4 \times D_3$, $\text{Aut}(T_{15,1,k}) = D_5 \times D_3$ for $k = 3, 4$ and $\text{Aut}(T_{m^2-m+1,1,m-1}) = \mathbb{Z}_{m^2-m+1}:\mathbb{Z}_6$ for $m = 3, 4$. Here we prove the following.

LEMMA 2.2. (a) D_n acts face-transitively on $T_{n,1,k}$ for all $n \geq 7$ and for all k .

(b) $\text{Aut}(T_{n,1,2}) = D_n$ for all $n \geq 9$.

(c) $D_{2m} \times D_{m+1} \leq \text{Aut}(T_{2m^2+2m,1,2m})$ for $m \geq 2$.

(d) $D_{m+1} \times D_{m-1} \leq \text{Aut}(T_{m^2-1,1,m-1})$ for $m \geq 4$.

(e) $D_{m+1} \times D_{m-1} \leq \text{Aut}(T_{m^2-1,1,m})$ for $m \geq 4$.

(f) $\mathbb{Z}_{m^2-m+1}:\mathbb{Z}_6 \leq \text{Aut}(T_{m^2-m+1,1,m-1})$ for $m \geq 3$.

Here $H \leq G$ means G has a subgroup isomorphic to H .

Proof. Let $\alpha_n, \beta_n: V(T_{n,1,k}) \rightarrow V(T_{n,1,k})$ be given by $\alpha_n(i) = i+1$ and $\beta_n(i) = n-i$ (modulo n). Let $A_{n,k,i} := \{i, i+1, i+k+1\}$ and $B_{n,k,i} := \{i, i+k, i+k+1\}$. Then $\alpha_n(A_{n,k,i}) = A_{n,k,i+1}$, $\alpha_n(B_{n,k,i}) = B_{n,k,i+1}$, $\beta_n(A_{n,k,i}) = B_{n,k,n-i-k-1}$ and $\beta_n(B_{n,k,i}) = A_{n,k,n-i-k-1}$. So, $\alpha_n, \beta_n \in \text{Aut}(T_{n,1,k})$. Clearly, the order of α_n is n , the order of β_n is 2 and $\beta_n \alpha_n \beta_n = \alpha_n^{-1}$. Thus, $\langle \alpha_n, \beta_n \rangle$ is isomorphic to D_n . Clearly, the action of $\langle \alpha_n, \beta_n \rangle$ on $T_{n,1,k}$ is transitive on the faces. This proves (a).

For $n \geq 11$, $G_4(\text{EG}(T_{n,1,2})) = C_n(1, 2, \dots, n)$. Therefore, $\langle \alpha_n, \beta_n \rangle \leq \text{Aut}(T_{n,1,2}) \leq \text{Aut}(G_4(\text{EG}(T_{n,1,2}))) = \text{Aut}(C_n(1, \dots, n)) = \langle \alpha_n, \beta_n \rangle$. Thus, $\text{Aut}(T_{n,1,2}) = \langle \alpha_n, \beta_n \rangle$ for $n \geq 11$. Since $G_2(\text{EG}(T_{10,1,2})) = C_{10}(1, 4, 7, 10, 3, 6, 9, 2, 5, 8)$, $\langle \alpha_{10}^3, \beta_{10} \rangle \leq \text{Aut}(T_{10,1,2}) \leq \text{Aut}(G_2(\text{EG}(T_{10,1,2}))) = \text{Aut}(C_{10}(1, 4, \dots, 5, 8)) = \langle \alpha_{10}^3, \beta_{10} \rangle$. Thus, $\text{Aut}(T_{10,1,2}) = \langle \alpha_{10}^3, \beta_{10} \rangle = \langle \alpha_{10}, \beta_{10} \rangle$. Observe that $\text{NEG}(T_{9,1,2}) = C_9(1, 5, 9, 4, 8, 3, 7, 2, 6)$. Therefore, $\langle \alpha_9^4, \beta_9 \rangle \leq \text{Aut}(T_{9,1,2}) \leq \text{Aut}(\text{NEG}(T_{9,1,2})) = \text{Aut}(C_9(1, 5, 9, 4, 8, 3, 7, 2, 6)) = \langle \alpha_9^4, \beta_9 \rangle$. Thus, $\text{Aut}(T_{9,1,2}) = \langle \alpha_9^4, \beta_9 \rangle = \langle \alpha_9, \beta_9 \rangle$. This proves (b).

Let $\alpha, \beta, \gamma, \delta: V(T_{2m^2+2m,1,2m}) \rightarrow V(T_{2m^2+2m,1,2m})$ be given by $\alpha(i) = i+m+1$, $\beta(i) = i+2m$, $\gamma(i) = (2m+1)i$ and $\delta(i) = (2m^2-1)i$ (i.e., $\alpha = \alpha_{2m^2+2m}^{m+1}$, $\beta = \alpha_{2m^2+2m}^{2m}$). Then

$\alpha, \beta, \gamma, \delta \in \text{Aut}(T_{2m^2+2m,1,2m})$, order of α is $2m$, order of β is $m+1$, order of γ is 2 , order of δ is 2 , $\alpha\beta = \beta\alpha$, $\alpha\gamma = \gamma\alpha$, $\beta\delta = \delta\beta$, $\gamma\delta = \delta\gamma = \beta_{2m^2+2m}$, $\delta\alpha\delta = \alpha^{-1}$, $\gamma\beta\gamma = \beta^{-1}$. Therefore, $\langle \alpha, \beta, \gamma, \delta \rangle = \langle \alpha, \delta \rangle \times \langle \beta, \gamma \rangle \cong D_{2m} \times D_{m+1}$. This proves (c).

Claim. If $m^2 \equiv 1 \pmod{n}$ then $\mu(i) = mi$ and $\nu(i) = (n-m)i$ define two distinct involutions (automorphisms of order 2) of $T_{n,1,k}$ for each $k \in \{m-1, m\}$.

Let $A_{n,k,i}$ and $B_{n,k,i}$ be as above. Then $\mu(A_{n,m-1,i}) = A_{n,m-1,mi}$, $\mu(B_{n,m-1,i}) = B_{n,m-1,mi-m+1}$, $\mu(A_{n,m,i}) = B_{n,m,mi}$ and $\mu(B_{n,m,i}) = A_{n,m,mi}$. Thus $\mu \in \text{Aut}(T_{n,1,k})$ and hence $\nu = \beta_n\mu = \mu\beta_n \in \text{Aut}(T_{n,1,k})$ for $k = m-1, m$. Since $\mu^{-1} = \mu$, $\nu^{-1} = \nu$ and $\mu\nu = \beta_n \neq$ the identity, $\mu \neq \nu$. This proves the claim.

For $k = m-1, m$, let $\alpha, \lambda, \mu, \nu: V(T_{m^2-1,1,k}) \rightarrow V(T_{m^2-1,1,k})$ be given by $\alpha(i) = i+m+1$, $\lambda(i) = i+m-1$, $\mu(i) = mi$ and $\nu(i) = (n-m)i$ (i.e., $\alpha = \alpha_{m^2-1}^{m+1}$, $\lambda = \alpha_{m^2-1}^{m-1}$). Then α and λ are automorphisms of $T_{m^2-1,1,k}$. Also, by the above claim, μ and ν are distinct automorphisms of $T_{m^2-1,1,k}$. Clearly, the orders of α , λ , μ and ν are $m-1$, $m+1$, 2 and 2 respectively. Again, $\alpha\mu = \mu\alpha$, $\alpha\lambda = \lambda\alpha$, $\lambda\nu = \nu\lambda$, $\mu\nu = \nu\mu$, $\nu\alpha\nu = \alpha^{-1}$ and $\mu\lambda\mu = \lambda^{-1}$. Thus, $\langle \alpha, \lambda, \mu, \nu \rangle = \langle \lambda, \mu \rangle \times \langle \alpha, \nu \rangle \cong D_{m+1} \times D_{m-1}$. This proves (d) and (e).

Let $\sigma: V(T_{m^2-m+1,1,m-1}) \rightarrow V(T_{m^2-m+1,1,m-1})$ be given by $\sigma(i) = mi$. Then $\sigma(A_{m^2-m+1,m-1,i}) = B_{m^2-m+1,m-1,mi}$ and $\sigma(B_{m^2-m+1,m-1,i}) = A_{m^2-m+1,m-1,mi-1}$. Thus, $\sigma \in \text{Aut}(T_{m^2-m+1,1,m-1})$. Since 6 is the smallest positive integer n for which $m^n - 1$ is divisible by $m^2 - m + 1$, the order of σ is 6 . Now, if $\rho = \alpha_{m^2-m+1}$ (i.e., $\rho(i) = i+1$) then ρ is an automorphism of order $m^2 - m + 1$ and $\sigma^{-1}\rho\sigma(i) = m((1-m)i+1) = i+m = \rho^m(i)$. Thus, $\langle \rho, \sigma \rangle = \mathbb{Z}_{m^2-m+1} : \mathbb{Z}_6$. This proves (f). \square

EXAMPLE 2: A series of weakly regular orientable combinatorial 2-manifolds of Euler characteristic 0 . For each $n \geq 4$ and each $k = 1, \dots, n-3$,

$$T_{n,2,k} = \{u_i u_{i+1} v_{i+1}, u_i v_i v_{i+1}, u_{i+k} u_{i+k+1} v_i, u_{i+k+1} v_i v_{i+1} : 1 \leq i \leq n\},$$

where $V(T_{n,2,k}) = \{u_1, \dots, u_n\} \cup \{v_1, \dots, v_n\}$. (Addition in the subscripts are modulo n .) Since $\text{lk}(u_i) = C_6(u_{i-1}, v_i, v_{i+1}, u_{i+1}, v_{n+i-k-1}, v_{n+i-k})$ and $\text{lk}(v_j) = C_6(v_{j-1}, u_{j-1}, u_j, v_{j+1}, u_{j+k+1}, u_{j+k})$, $T_{n,2,k}$ is a degree-regular combinatorial 2-manifold on $2n$ vertices. Clearly, $T_{n,2,k}$ triangulates the torus and hence it is orientable. If $\alpha, \beta: V(T_{n,2,k}) \rightarrow V(T_{n,2,k})$ are the maps given by $\alpha(u_i) = u_{i+1}$, $\alpha(v_i) = v_{i+1}$, $\beta(u_i) = v_i$, $\beta(v_i) = u_{i+k}$ for $1 \leq i \leq n$, then α and β are automorphisms of $T_{n,2,k}$ and hence $\langle \alpha, \beta \rangle$ is a subgroup of $\text{Aut}(T_{n,2,k})$. Clearly, $\langle \alpha, \beta \rangle$ acts transitively on vertices. Thus $T_{n,2,k}$ is weakly regular.

LEMMA 2.3. Let $T_{n,1,j}$ and $T_{m,2,k}$ be as in Examples 1, 2. We have the following:

- (a) If n and k are relatively prime or n and $k+2$ are relatively prime then $T_{n,2,k}$ is isomorphic to $T_{2n,1,j}$ for some j .
- (b) $T_{6,2,2} \not\cong T_{12,1,i}$ for all i .
- (c) $T_{8,2,4} \cong T_{8,2,2} \cong T_{16,1,i}$ for all i .

Proof. If $(n, k) = 1$ and $k \leq n-3$, then there exists $p \in \{1, \dots, n-2\}$ such that $pk \equiv 1 \pmod{n}$. Since $k \leq n-3$ and $n \geq 4$, $2p \neq n, n-1$. ($2p = n$ implies $2 \equiv 2pk \equiv 0 \pmod{n}$, a contradiction. $2p = n-1$ implies $2 \equiv 2pk \equiv n-k$. This implies $k \equiv n-2 \pmod{n}$ and hence $k = n-2$.) Let $\varphi: V(T_{n,2,k}) \rightarrow \{1, \dots, 2n\}$ be given by $\varphi(u_i) = 1 + 2p(i-1)$ and $\varphi(v_i) = 2 + 2p(i-1)$ (modulo $2n$). Since $(n, p) = 1$, φ is a bijection. Now, $\varphi(u_i u_{i+1} v_{i+1}) = \{1+2(i-1)p, 1+2ip, 2+2ip\}$, $\varphi(u_i v_i v_{i+1}) = \{1+2(i-1)p, 2+2(i-1)p, 2+2ip\}$, $\varphi(u_{i+k} u_{i+k+1} v_i) =$

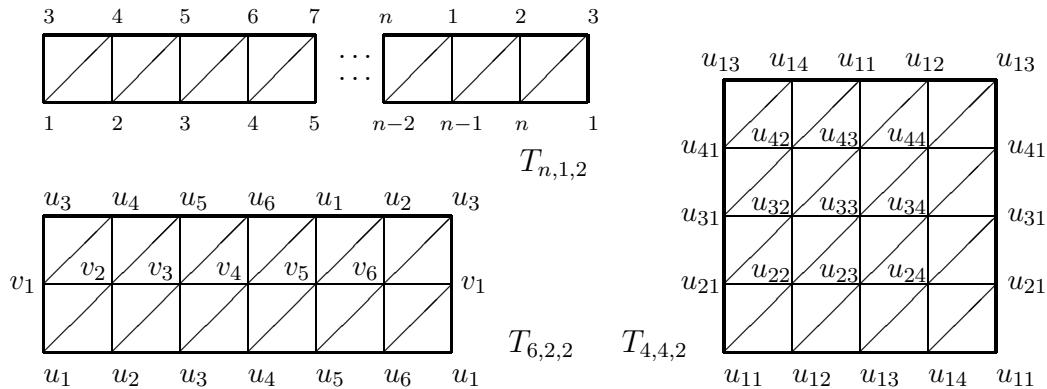
$\{1 + 2(i-1)p + 2, 1 + 2ip + 2, 2 + 2(i-1)p\} = \{2(i-1)p + 2, 2(i-1)p + 3, 2ip + 3\}$,
 $\varphi(u_{i+k+1}v_iv_{i+1}) = \{2ip + 3, 2(i-1)p + 2, 2 + 2ip\} = \{2(i-1)p + 2, 2ip + 2, 2ip + 3\} \in T_{2n,1,2p}$.
This shows that $\varphi: T_{n,2,k} \rightarrow T_{2n,1,2p}$ is an isomorphism.

If $(n, k+2) = 1$ then assume that $(n, k) \neq 1$ (otherwise there is nothing to prove). Let $p(k+2) \equiv 1 \pmod{n}$. Observe that $2p \neq n, n-1$. ($2p = n$ implies $2 \equiv 2p(k+2) \equiv 0 \pmod{n}$, a contradiction. $2p = n-1$ implies $2 \equiv 2p(k+2) \equiv (n-1)(k+2) \equiv n-k-2$. This implies $k \equiv n-4 \pmod{n}$ and hence $k = n-4$. Since $(n, k) \neq 1$, n and k are even and hence $(n, k+2) \neq 1$, a contradiction.) Let $\psi: V(T_{n,2,k}) \rightarrow \{1, \dots, 2n\}$ be given by $\psi(u_i) = 1 + 2p(i-1)$ and $\psi(v_i) = 2 + 2p(i-2)$ (modulo $2n$). Since $(n, p) = 1$, ψ is a bijection. Similar argument as before shows that $\psi: T_{n,2,k} \rightarrow T_{2n,1,2p-1}$ is an isomorphism. This proves (a).

Since $G_4(\text{EG}(T_{6,2,2})) = 3K_4$ and $G_4(\text{EG}(T_{12,1,i})) = C_{12}$ for $i = 2, 3$, $T_{6,2,2} \not\cong T_{12,1,i}$ for $i = 2, 3$.

Now, $G_3(\text{EG}(T_{12,1,4})) = 4C_3$ (with edges $\{i, i+4\}$, $1 \leq i \leq 12$). So, $G_3(\text{EG}(T_{12,1,4}))$ is a subgraph of $\text{EG}(T_{12,1,4})$. Whereas, $G_3(\text{EG}(T_{6,2,2})) = 4C_3$ (with edges $\{u_i, u_j\}$, $\{v_i, v_j\}$, where $i-j \equiv 0 \pmod{2}$). So, edges of $G_3(\text{EG}(T_{6,2,2}))$ are not in $\text{EG}(T_{6,2,2})$. Thus, $T_{6,2,2} \not\cong T_{12,1,4}$. This proves (b).

The map $u_i \mapsto u_i$, $v_i \mapsto v_{i+3}$ defines an isomorphism from $T_{8,2,2}$ to $T_{8,2,4}$. (As usual, addition in the subscripts are modulo 8.) Observe that $G_1(\text{EG}(T_{8,2,2}))$ is a null graph, whereas $G_1(\text{EG}(T_{16,1,2})) = 2C_8$, $G_1(\text{EG}(T_{16,1,3})) = 2C_8$, $G_1(\text{EG}(T_{16,1,6})) = 4C_4$. Thus $T_{8,2,2} \not\cong T_{16,1,j}$ for $j = 2, 3$ and 6. Therefore, by Lemma 2.1 (g), $T_{8,2,2} \not\cong T_{16,1,i}$ for all i . This proves (c). \square



EXAMPLE 3: Some more weakly regular orientable combinatorial 2-manifolds of Euler characteristic 0. For $n, m \geq 3$ and $k = 0, \dots, n-1$,

$$\begin{aligned} T_{n,m,k} &= \{u_{i,j}u_{i,j+1}u_{i+1,j+1}, u_{i,j}u_{i+1,j}u_{i+1,j+1} : 1 \leq i \leq m-1, 1 \leq j \leq n\} \\ &\cup \{u_{m,j}u_{m,j+1}u_{1,j+k+1}, u_{m,j}u_{1,j+k}u_{1,j+k+1} : 1 \leq j \leq n\}, \end{aligned}$$

where $V(T_{n,m,k}) = \{u_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$. (Addition in the second subscripts are modulo n .) Clearly, $T_{n,m,k}$ triangulates the torus and hence it is an orientable combinatorial 2-manifold on mn vertices. Since the degree of each vertex is 6, $T_{n,m,k}$ is degree-regular. ($T_{n,m,0}$ was earlier defined in [5] as $A_{m,n}$.) If $\sigma, \gamma: V(T_{n,m,k}) \rightarrow V(T_{n,m,k})$ are the maps given by $\gamma(u_{i,j}) = u_{i+1,j}$ for $1 \leq i \leq m-1$, $\gamma(u_{m,j}) = u_{1,j+k}$ and $\sigma(u_{i,j}) = u_{i,j+1}$ then σ and γ are automorphisms of $T_{n,m,k}$ and hence $\langle \sigma, \gamma \rangle$ is a subgroup of $\text{Aut}(T_{n,m,k})$. For any vertex $u_{i,j}$, $\sigma^{j-1}\gamma^{i-1}(u_{1,1}) = u_{i,j}$. This implies that the action of $\langle \sigma, \gamma \rangle$ is vertex-transitive. Thus $T_{n,m,k}$ is weakly regular.

LEMMA 2.4. Let $T_{n,m,k}$ be as in Examples 1, 2, 3. Then

- (a) If n and k are relatively prime or n and $k+m$ are relatively prime then $T_{n,m,k}$ is isomorphic to $T_{nm,1,j}$ for some j .
- (b) $T_{4,4,2} \cong T_{8,2,2}$.
- (c) $T_{16,1,k} \not\cong T_{4,4,0} \not\cong T_{8,2,j}$ for all k and j .

Proof. Since $(n, k) = 1$, there exists $p \in \{1, \dots, n-1\}$ such that $pk \equiv 1 \pmod{n}$. Let $\varphi: V(T_{n,m,k}) \rightarrow \{1, \dots, mn\}$ be given by $\varphi(u_{i,j}) = i + mp(j-1) \pmod{mn}$. Since $(n, p) = 1$, φ is a bijection. By the similar argument as in the proof of Lemma 2.3, $\varphi: T_{n,m,k} \rightarrow T_{mn,1,mp}$ is an isomorphism.

Let $(n, k+m) = 1$ where $m \geq 3$. Let $p \in \{1, \dots, n-1\}$ be such that $p(k+m) \equiv 1 \pmod{n}$. Let $\psi: V(T_{n,2,k}) \rightarrow \{1, \dots, mn\}$ be given by $\psi(u_{i,j}) = i + mp(j-i) \pmod{mn}$. Since $(n, p) = 1$, ψ is a bijection. Now, $\psi(u_{i,j}u_{i,j+1}u_{i+1,j+1}) = \{i + mp(j-i), i + mp(j+1-i), i + 1 + mp(j-i)\}$, $\psi(u_{i,j}u_{i+1,j}u_{i+1,j+1}) = \{i + mp(j-i), i + 1 + mp(j-i-1), i + 1 + mp(j-i)\}$, $\psi(u_{m,j}u_{m,j+1}u_{1,j+k+1}) = \{m + mp(j-m), m + mp(j+1-m), 1 + mp(j+k)\} = \{m + mp(j-m), m + mp(j+1-m), 1 + m + mp(j-m)\}$, $\psi(u_{m,j}u_{1,j+k}u_{1,j+k+1}) = \{m + mp(j-m), 1 + mp(j+k-1), 1 + mp(j+k)\} = \{m + mp(j-m), 1 + m + mp(j-m-1), 1 + m + mp(j-m)\} \in T_{mn,1,mp-1}$. (Clearly, if mn is even then $mn/2 - 1 \neq mp - 1 \neq mn/2$ and if mn is odd then $mp - 1 \neq (mn-1)/2$.) Thus $\psi: T_{n,m,k} \rightarrow T_{mn,1,mp-1}$ is an isomorphism. This proves (a).

The map $u_{i,1} \mapsto u_i$, $u_{i,3} \mapsto u_{i+4}$, $u_{i,2} \mapsto v_i$ and $u_{i,4} \mapsto v_{i+4}$ for $1 \leq i \leq 4$ defines an isomorphism from $T_{4,4,2}$ to $T_{8,2,2}$. This proves (b).

Since $G_1(\text{EG}(T_{16,1,2})) = 2C_8$, $G_1(\text{EG}(T_{16,1,3})) = 2C_8$, $G_1(\text{EG}(T_{16,1,6})) = 4C_4$ and $G_1(\text{EG}(T_{4,4,0}))$ is a null graph, $T_{4,4,0}$ is not isomorphic to any of $T_{16,1,2}$, $T_{16,1,3}$ and $T_{16,1,6}$. Again $G_4(\text{EG}(T_{4,4,0}))$ is a null graph whereas $G_4(\text{EG}(T_{8,2,2})) = 8K_2$. Thus $T_{4,4,0} \not\cong T_{8,2,2}$. (c) now follows from parts (a), (c) of Lemma 2.3 and part (g) of Lemma 2.1. \square

Now, we will present three series of degree-regular triangulations of the Klein bottle. Among these, $B_{m,n}$ were defined earlier in [5]. The smallest among $Q_{m,n}$'s, namely $Q_{5,2}$, also defined in [5] as Q .

EXAMPLE 4: A series of degree-regular non-orientable combinatorial 2-manifolds of Euler characteristic 0. For $m, n \geq 3$,

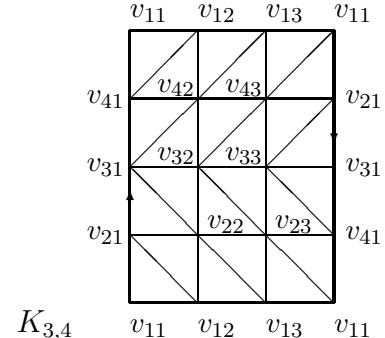
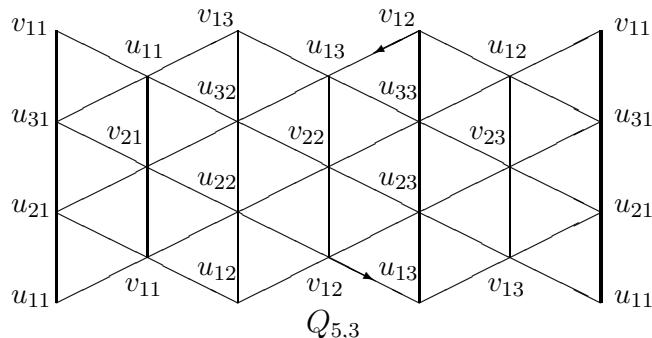
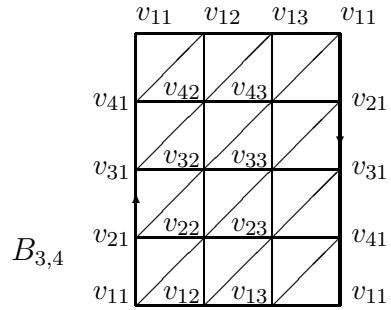
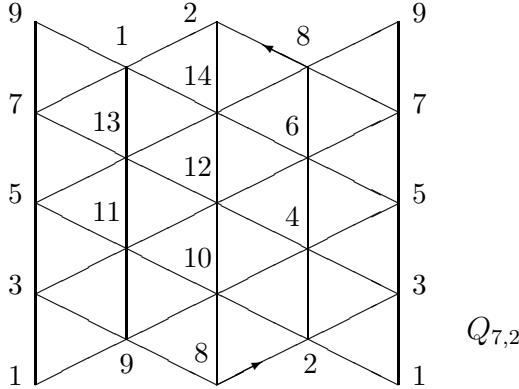
$$\begin{aligned} B_{m,n} = & \{v_{i,j}v_{i+1,j}v_{i+1,j+1}, v_{i,j}v_{i,j+1}v_{i+1,j+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\} \\ & \cup \{v_{i,m}v_{n+2-i,1}v_{n+1-i,1}, v_{i,m}v_{i+1,m}v_{n+1-i,1} : 1 \leq i \leq n\}, \end{aligned}$$

where $V(B_{m,n}) = \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$. (Addition in the first subscripts are modulo n .) Clearly, $B_{m,n}$ triangulates the Klein bottle and hence it is a non-orientable combinatorial 2-manifold. Since the degree of each vertex is 6, $B_{m,n}$ is degree-regular.

EXAMPLE 5: A series of degree-regular non-orientable combinatorial 2-manifolds of Euler characteristic 0. For $m \geq 3$ and $n \geq 2$,

$$\begin{aligned} K_{m,2n} = & \{v_{i,j}v_{i,j+1}v_{i+1,j}, v_{i,j+1}v_{i+1,j}v_{i+1,j+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\} \\ & \cup \{v_{i,m}v_{i+1,m}v_{2n+2-i,1}, v_{i+1,m}v_{2n+2-i,1}v_{2n+1-i,1} : 1 \leq i \leq n\} \\ & \cup \{v_{i,j}v_{i+1,j}v_{i+1,j+1}, v_{i,j}v_{i,j+1}v_{i+1,j+1} : n+1 \leq i \leq 2n, 1 \leq j \leq m-1\} \\ & \cup \{v_{i,m}v_{i+1,m}v_{2n+1-i,1}, v_{i,m}v_{2n+2-i,1}v_{2n+1-i,1} : n+1 \leq i \leq 2n\}, \end{aligned}$$

where $V(K_{m,2n}) = \{v_{i,j} : 1 \leq i \leq 2n, 1 \leq j \leq m\}$. (Addition in the first subscripts are modulo $2n$.) Clearly, $K_{m,2n}$ triangulates the Klein bottle and hence it is a non-orientable combinatorial 2-manifold. Since the degree of each vertex is 6, $K_{m,2n}$ is degree-regular.



EXAMPLE 6: A series of weakly regular non-orientable combinatorial 2-manifolds of Euler characteristic 0. For each $m \geq 2$,

$$Q_{2m+1,2} = \{\{i, i+1, i+2\}, \{i, i+2, i+2m+2\} : 1 \leq i \leq 4m+2\},$$

where $V(Q_{2m+1,2}) = \{1, \dots, 4m+2\}$. (Addition modulo $4m+2$.) Clearly, $Q_{2m+1,2}$ triangulates the Klein bottle and hence it is a non-orientable combinatorial 2-manifold. Since \mathbb{Z}_{4m+2} acts transitively (by addition) on vertices, $Q_{2m+1,2}$ is weakly regular.

More generally, for each $n \geq 2$ we define the following.

EXAMPLE 7: A series of degree-regular non-orientable combinatorial 2-manifolds of Euler characteristic 0. For $m, n \geq 2$,

$$\begin{aligned} Q_{2m+1,n} = & \{u_{i,j}u_{i+1,j}v_{i,j}, u_{i,j+1}u_{i+1,j+1}v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \\ & \cup \{v_{i,j}v_{i+1,j}u_{i+1,j}, v_{i,j}v_{i+1,j}u_{i+1,j+1} : 1 \leq i \leq m-1, 1 \leq j \leq n\} \\ & \cup \{u_{m+1,j}u_{1,n+2-j}v_{1,n+2-j}, u_{m+1,j+1}u_{1,n+2-j}v_{1,n+1-j}, \\ & \quad u_{m+1,j}u_{1,n+2-j}v_{m,j}, u_{m+1,j+1}u_{1,n+2-j}v_{m,j} : 1 \leq j \leq n\}, \end{aligned}$$

where $V(Q_{2m+1,n}) = \{u_{i,j} : 1 \leq i \leq m+1, 1 \leq j \leq n\} \cup \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$. (Addition in the second subscripts are modulo n .) Clearly, $Q_{2m+1,n}$ triangulates the Klein bottle and hence it is a non-orientable combinatorial 2-manifold. Since the degree of each vertex is 6, $Q_{2m+1,n}$ is degree-regular.

For $n \geq 3$ there are two induced 3-cycles (induced subcomplexes which are 3-cycles) through $v_{2,1}$ in $Q_{5,n}$, namely, $C_3(v_{2,1}, u_{1,1}, u_{2,1})$ and $C_3(v_{2,1}, u_{3,1}, v_{1,1})$. But there is no induced 3-cycle through $v_{2,n}$. So, there does not exist any automorphism of $Q_{5,n}$ which sends $v_{2,1}$ to $v_{2,n}$. Thus, $Q_{5,n}$ is not weakly regular for $n \geq 3$.

Observe that $G_3(\text{EG}(Q_{7,2t-1})) = C_7(u_{1,1}, u_{3,1}, v_{1,1}, v_{3,1}, u_{2,1}, u_{4,1}, v_{2,1}) \sqcup C_7(u_{1,t+1}, u_{3,t+1}, v_{1,t}, v_{3,t}, u_{2,t+1}, u_{4,t+1}, v_{2,t})$ and $G_3(\text{EG}(Q_{7,2t-2})) = C_7(u_{1,1}, u_{3,1}, v_{1,1}, v_{3,1}, u_{2,1}, u_{4,1}, v_{2,1}) \sqcup C_7(u_{1,t}, u_{3,t}, v_{1,t}, v_{3,t}, u_{2,t}, u_{4,t}, v_{2,t})$ for $t \geq 2$. So, for $n \geq 2$, $G_3(\text{EG}(Q_{7,n}))$ consists of two disjoint 7-cycles. This implies that $G_3(\text{EG}(Q_{7,n}))$ is not regular for $n \geq 3$ and hence $Q_{7,n}$ is not weakly regular for $n \geq 3$.

For each $m \geq 2$, there are exactly two $(2m+1)$ -cycles (namely, $C_{2m+1}(u_{1,1}, \dots, u_{m+1,1}, v_{1,1}, \dots, v_{m,1})$, $C_{2m+1}(u_{1,t+1}, \dots, u_{m+1,t+1}, v_{1,t}, \dots, v_{m,t})$ in $Q_{2m+1,2t-1}$ and $C_{2m+1}(u_{1,1}, \dots, u_{m+1,1}, v_{1,1}, \dots, v_{m,1})$, $C_{2m+1}(u_{1,t+1}, \dots, u_{m+1,t+1}, v_{1,t+1}, \dots, v_{m,t+1})$ in $Q_{2m+1,2t}$) each of which is the boundary of a $(2m+1)$ -vertex Möbius strip. In other words, there are exactly two $(2m+1)$ -cycle, say C_1 and C_2 , such that $|Q_{2m+1,n}| \setminus |C_i|$ is union of two disjoint open Möbius strips for $i = 1, 2$. Thus, if $n \geq 3$ and u is a vertex in C_1 and v is a vertex outside $C_1 \cup C_2$ then there does not exist any automorphism of $Q_{2m+1,n}$ which sends u to v . So, $Q_{2m+1,n}$ is not weakly regular for all $m \geq 2$ and $n \geq 3$.

LEMMA 2.5. *Let $B_{m,n}$, $K_{m,2n}$ and $Q_{2m+1,n}$ be as above. We have the following :*

- (a) $B_{3,4} \not\cong B_{4,3} \not\cong K_{3,4} \not\cong B_{3,4}$.
- (b) $B_{3,5} \not\cong B_{5,3} \not\cong Q_{5,3} \not\cong B_{3,5}$.
- (c) *None of $B_{3,4}$, $B_{4,3}$, $K_{3,4}$, $B_{3,5}$, $B_{5,3}$, $Q_{5,3}$ are weakly regular.*

Proof. Observe that $G_4(\text{EG}(B_{3,4})) = C_3(v_{11}, v_{42}, v_{23}) \cup C_3(v_{41}, v_{12}, v_{13}) \cup C_3(v_{31}, v_{22}, v_{43}) \cup C_3(v_{21}, v_{32}, v_{33})$, $G_4(\text{EG}(B_{4,3})) = C_8(v_{11}, v_{32}, v_{23}, v_{14}, v_{21}, v_{12}, v_{33}, v_{24}) \cup K_4(\{v_{31}, v_{22}, v_{13}, v_{34}\})$ and $G_4(\text{EG}(K_{3,4})) = K_4(\{v_{11}, v_{22}, v_{33}, v_{42}\}) \cup K_4(\{v_{12}, v_{23}, v_{31}, v_{43}\}) \cup K_4(\{v_{13}, v_{21}, v_{32}, v_{41}\})$. These proves (a) (since $M \cong N$ implies $G_4(\text{EG}(M)) \cong G_4(\text{EG}(N))$).

It is easy to see the following: (i) $G_4(\text{EG}(B_{5,3})) = C_5(v_{21}, v_{12}, v_{33}, v_{24}, v_{15}) \cup C_{10}(v_{11}, v_{32}, v_{23}, v_{14}, v_{35}, v_{31}, v_{22}, v_{13}, v_{34}, v_{25})$, (ii) $G_4(\text{EG}(B_{3,5}))$ is $C_3(v_{11}, v_{52}, v_{23}) \cup C_3(v_{31}, v_{32}, v_{43}) \cup C_3(v_{21}, v_{42}, v_{33}) \cup C_3(v_{51}, v_{12}, v_{13})$ together with the three isolated vertices v_{41}, v_{22}, v_{53} and (iii) $G_4(\text{EG}(Q_{5,3}))$ is $C_5(u_{11}, u_{21}, u_{31}, v_{11}, v_{21}) \cup C_5(u_{13}, u_{23}, u_{33}, v_{12}, v_{22})$ together with five isolated vertices. These prove (b).

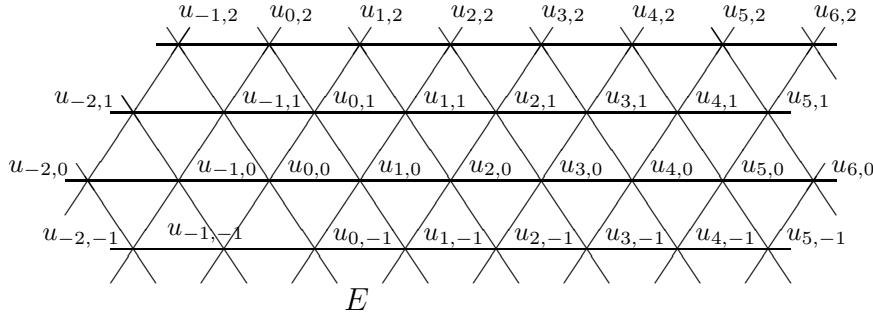
If M is weakly regular then $\text{Aut}(M)$ acts vertex-transitively on $G_n(\text{EG}(M))$ for all $n \geq 0$. Since $G_4(\text{EG}(B_{4,3})) = C_8 \sqcup K_4$, no group can act vertex-transitively on $G_4(\text{EG}(B_{4,3}))$. So, $B_{4,3}$ is not weakly regular. Similarly, $B_{5,3}$ is not weakly regular. Since $G_4(\text{EG}(B_{3,5}))$ and $G_4(\text{EG}(Q_{5,3}))$ are not regular graphs, $B_{3,5}$ and $Q_{5,3}$ are not weakly regular.

Observe that $G_3(\text{EG}(B_{3,4})) \cap \text{EG}(B_{3,4}) = C_6(v_{11}, v_{12}, v_{23}, v_{41}, v_{42}, v_{13}) \cup C_6(v_{31}, v_{32}, v_{43}, v_{21}, v_{22}, v_{33})$. If possible let there be $\sigma \in \text{Aut}(B_{3,4})$ such that $\sigma(v_{11}) = v_{12}$. Since $\text{Aut}(B_{3,4})$ acts vertex-transitively on the graph $G_3(\text{EG}(B_{3,4})) \cap \text{EG}(B_{3,4})$, $\sigma(v_{12}) = v_{11}$ or v_{23} . In the first case, $\sigma(v_{41}) = v_{42}$ and hence $\sigma(v_{11}v_{12}v_{41}) = v_{11}v_{12}v_{42}$. But $v_{11}v_{12}v_{41}$ is a face, whereas $v_{11}v_{12}v_{42}$ is not a face, a contradiction. In the second case, $(v_{11}, v_{12}, v_{23}, v_{41}, v_{42}, v_{13})$ is a cycle in (the permutation) σ . Then $\sigma(v_{11}v_{12}v_{41}) = v_{12}v_{23}v_{42}$. But $v_{11}v_{12}v_{41}$ is a face,

whereas $v_{12}v_{23}v_{42}$ is not a face, a contradiction. So, there is no automorphism which maps v_{11} to v_{12} . Therefore, $B_{3,4}$ is not weakly regular.

If possible let there be $\tau \in \text{Aut}(K_{3,4})$ such that $\tau(v_{11}) = v_{21}$. Since $G_3(\text{EG}(K_{3,4})) = C_3(v_{11}, v_{12}, v_{13}) \cup C_3(v_{21}, v_{23}, v_{42}) \cup C_3(v_{31}, v_{32}, v_{33}) \cup C_3(v_{41}, v_{22}, v_{43})$, $\tau(v_{12}) = v_{23}$ or v_{42} . In either case τ maps the edge $v_{11}v_{12}$ of $K_{3,4}$ to a non-edge of $K_{3,4}$, a contradiction. So, there is no automorphism which maps v_{11} to v_{21} . Thus, $K_{3,4}$ is not weakly regular. \square

EXAMPLE 8: A triangulation (E) of the plane \mathbb{R}^2 . The vertex-set $V(E) = \{u_{m,2n} = (m, n\sqrt{3}), u_{m,2n-1} = (m + \frac{1}{2}, \frac{(2n-1)\sqrt{3}}{2}) : m, n \in \mathbb{Z}\}$ and the faces are $\{u_{m,2n}u_{m+1,2n}u_{m,2n+1}, u_{m+1,2n}u_{m,2n+1}u_{m+1,2n+1}, u_{m,2n-1}u_{m+1,2n-1}u_{m+1,2n}, u_{m,2n-1}u_{m,2n}u_{m+1,2n} : m, n \in \mathbb{Z}\}$. The group H of translations generated by $\alpha_1 : u \mapsto u + u_{1,0}$ and $\alpha_2 : u \mapsto u + u_{0,1}$ is a subgroup of $\text{Aut}(E)$. Clearly, H acts transitively on $V(E)$. The stabilizer of any vertex u in $\text{Aut}(E)$ is isomorphic to the dihedral group D_6 (of order 12) which acts transitively on the set of flags containing u . So, E is combinatorially regular. Let G_0 denote the stabilizer of $u_{0,0}$. Since H acts transitively on $V(E)$, $\text{Aut}(E) = \langle H, G_0 \rangle$. This implies that if $\sigma \in \text{Aut}(E)$ has no fixed element in E (vertex, edge or face) then either $\sigma \in H \setminus \{\text{Id}\}$ or is a glide reflection (i.e., an automorphism of the form $t_a \circ r_l$, where $r_l \in \text{Aut}(E)$ is the reflection about a line l through some vertex and of slope a multiple of $\pi/6$ and $t_a \in H$ is the translation by a nonzero vector a parallel to l).



3 Proofs.

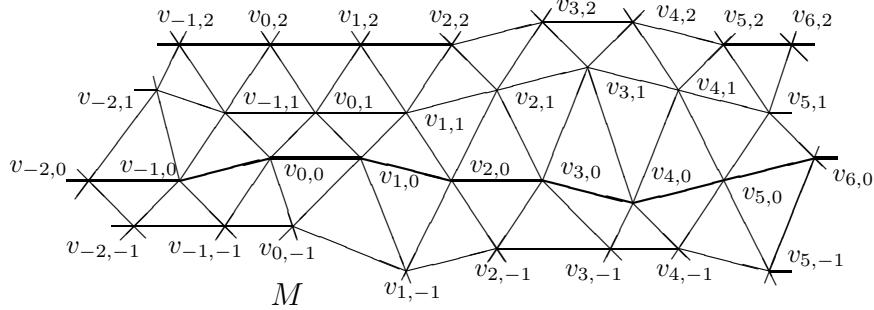
LEMMA 3.1. *There is no triangulation of the closed 2-disk such that (i) the degree of each vertex (except one) in the boundary is 4 and (ii) the degree of each interior vertex is 6.*

Proof. If possible let there be a triangulation K of the closed 2-disk on $m+n+1$ vertices with n interior vertices such that the degree of each interior vertex is 6, the degree of one vertex in the boundary is k (≥ 2) and the degree of each of the remaining m vertices in the boundary is 4. Then $f_0(K) = n+m+1$, $f_1(K) = \frac{6n+4m+k}{2}$ and $f_2(K) = \frac{6n+3m+k-1}{3}$. Therefore, $1 = \chi(K) = f_0(K) - f_1(K) + f_2(K) = n+m+1 - (3n+2m+k/2) + (2n+m+(k-1)/3)$. This implies that $k = -2$, a contradiction. This proves the lemma. \square

LEMMA 3.2. *Let E be as in Example 8 and let M be a triangulation of the plane \mathbb{R}^2 . If the degree of each vertex of M is 6 then M is isomorphic to E .*

Proof. Choose an edge, say $v_{0,0}v_{1,0}$. Then there exists a unique vertex, say $v_{2,0}$, in $\text{lk}(v_{1,0})$ such that each side of the segment $v_{0,0}v_{1,0}v_{2,0}$ (union of two line segments) contains three faces from $\text{st}(v_{1,0})$ (i.e., $\text{lk}(v_{1,0})$ is of the form $C_6(v_{0,0}, x, y, v_{2,0}, z, w)$). Now, given $v_{1,0}$ and $v_{2,0}$ there exists unique vertex $v_{3,0}$ in $\text{lk}(v_{2,0})$ such that each side of the segment $v_{1,0}v_{2,0}v_{3,0}$

contains three faces from $\text{st}(v_{2,0})$. Similarly, given $v_{1,0}$ and $v_{0,0}$ there exists unique vertex $v_{-1,0}$ in $\text{lk}(v_{0,0})$ such that each side of the segment $v_{1,0}v_{0,0}v_{-1,0}$ contains three faces from $\text{st}(v_{0,0})$. Continuing this way we get vertices $v_{i,0}$, $i \in \mathbb{Z}$, such that each side of the segment $v_{i-1,0}v_{i,0}v_{i+1,0}$ contains three faces from $\text{st}(v_{i,0})$. Because of Lemma 3.1, all these vertices are distinct. So, we get a triangulation of a line (see the figure).



Let $\text{lk}(v_{1,0}) = C_6(v_{0,0}, v_{0,1}, v_{1,1}, v_{2,0}, v_{1,-1}, v_{0,-1})$. By the same argument as above, there exists a unique vertex, say $v_{2,1}$, in $\text{lk}(v_{1,1})$ such that each side of the segment $v_{0,1}v_{1,1}v_{2,1}$ contains three faces from $\text{st}(v_{1,1})$. This implies that $\text{lk}(v_{2,0})$ is of the form $C_6(v_{1,0}, v_{1,1}, v_{2,1}, v_{3,0}, x, v_{1,-1})$. If we continue this way we get vertices $v_{i,1}$, $i \in \mathbb{Z}$, such that each side of the segment $v_{i-1,1}v_{i,1}v_{i+1,1}$ contains three faces from $\text{st}(v_{i,1})$ and $v_{i,0}v_{i+1,0}v_{i,1}$, $v_{i+1,0}v_{i,1}v_{i+1,1}$ are faces for all $i \in \mathbb{Z}$.

Similarly, we get: (i) vertices $v_{i,2}$, $i \in \mathbb{Z}$, such that each side of the segment $v_{i-1,2}v_{i,2}v_{i+1,2}$ contains three faces from $\text{st}(v_{i,2})$ and $v_{i,1}v_{i+1,1}v_{i+1,2}$, $v_{i,1}v_{i,2}v_{i+1,2}$ are faces for all $i \in \mathbb{Z}$. (ii) vertices $v_{i,-1}$, $i \in \mathbb{Z}$, such that each side of the segment $v_{i-1,-1}v_{i,-1}v_{i+1,-1}$ contains three faces from $\text{st}(v_{i,-1})$ and $v_{i,-1}v_{i+1,-1}v_{i+1,0}$, $v_{i,-1}v_{i,0}v_{i+1,0}$ are faces for all $i \in \mathbb{Z}$.

Continuing this way we get vertices $v_{i,j}$, $i, j \in \mathbb{Z}$, of M such that each side of the segment $v_{i-1,j}v_{i,j}v_{i+1,j}$ contains three faces from $\text{st}(v_{i,j})$ and $v_{i,2k}v_{i+1,2k}v_{i,2k+1}$, $v_{i+1,2k}v_{i,2k+1}v_{i+1,2k+1}$, $v_{i,2k+1}v_{i+1,2k+1}v_{i+1,2k+2}$, $v_{i,2k+1}v_{i,2k+2}v_{i+1,2k+2}$ are faces for all $i, j, k \in \mathbb{Z}$. Since M is connected $\{v_{i,j} : i, j \in \mathbb{Z}\}$ is the vertex-set of M . Then $\varphi: V(M) \rightarrow V(E)$, given by $\varphi(v_{i,j}) = u_{i,j}$, is an isomorphism. This proves the lemma. \square

Proof of Theorem 1. Let K be a degree-regular triangulation of the torus. Since \mathbb{R}^2 is the universal cover of the torus, there exists a triangulation M of \mathbb{R}^2 and a simplicial covering map $\eta: M \rightarrow K$ (cf. [12, Page 144]). Since the degree of each vertex in K is 6, the degree of each vertex in M is 6. Because of Lemma 3.2, we may assume that $M = E$.

Let Γ be the group of covering transformations. Then $|K| = |E|/\Gamma$. For $\sigma \in \Gamma$, $\eta \circ \sigma = \eta$. So, σ maps the geometric carrier of a simplex to the geometric carrier of a simplex. This implies that σ induces an automorphism σ of E . Thus, we can identify Γ with a subgroup of $\text{Aut}(E)$. So, K is a quotient of E by a subgroup Γ of $\text{Aut}(E)$, where Γ has no fixed element (vertex, edge or face). Hence Γ consists of translations and glide reflections. Since $K = E/\Gamma$ is orientable, Γ does not contain any glide reflection. Thus $\Gamma \leq H$ (the group of translations). Now H is commutative. So, Γ is a normal subgroup of H . Since H acts transitively on $V(E)$, H/Γ acts transitively on the vertices of E/Γ . Thus, K is weakly regular. \square

LEMMA 3.3. *For a prime $n \geq 7$, if M is an n -vertex weakly regular combinatorial 2-manifold of Euler characteristic 0 then M is isomorphic to $T_{n,1,k}$ for some k .*

Proof. Since M is weakly regular, it is degree-regular. Let d be the degree of each vertex. Then $nd = 2f_1(M)$ and $n - f_1(M) + f_2(M) = \chi(M) = 0$. Since each edge is in two triangles, $2f_1(M) = 3f_2(M)$. These imply that $d = 6$.

Let $G = \text{Aut}(M)$. Then G is isomorphic to a subgroup of the permutation group S_n and hence G is a finite group. Fix a vertex u of M . Let H be the stabilizer of u in G . Since M is weakly regular, the orbit of u under the action of G contains all the n vertices and hence the index of H in G is n . Thus, n divides the order of G . Since n is prime, G has an element, say τ , of order n .

Let v be a vertex in M such that $\tau(v) \neq v$. Then $V(M) = \{v, \tau(v), \dots, \tau^{n-1}(v)\}$. Choose an edge e containing v . Let $e = v\tau^k(v)$. Then $\sigma = \tau^k$ is again an automorphism of order n and $V(M) = \{v_0 = v, v_1 = \sigma(v), \dots, v_{n-1} = \sigma^{n-1}(v)\}$. For each $i = 0, \dots, n-1$, σ^i is an automorphism. Thus, v_kv_l is an edge implies $v_{k+i}v_{l+i}$ is an edge and $v_kv_lv_j$ is a face implies $v_{k+i}v_{l+i}v_{j+i}$ is a face for each i . Since v_0v_1 is an edge, v_iv_{i+1} is an edge for each i . (Addition in the subscripts are modulo n .)

Claim. $v_0v_1v_2$, $v_0v_1v_{n-1}$ and $v_0v_1v_{\frac{n+1}{2}}$ are not faces.

If $v_0v_1v_2$ is a face then $v_{n-2}v_{n-1}v_0$, $v_{n-1}v_0v_1$ are faces. Let $v_0v_2v_i$ ($\neq v_0v_1v_2$) be the second face containing v_0v_2 . Then $v_{n-2}v_0v_{i-2}$ is a face and hence $\text{lk}(v_0) = C_6(v_i, v_2, v_1, v_{n-1}, v_{n-2}, v_{i-2})$. Then $v_{i-2}v_iv_0$ is a face and hence $v_0v_2v_{n-i+2}$ is a face. This implies that $n - i + 2 = 1$ or i . Since $i \neq 1$, $n - i + 2 = i$. Then $n = 2i - 2$. This is not possible since n is a prime. So, $v_0v_1v_2$ is not a face and hence $v_0v_1v_{n-1}$ is not a face.

Let $c = \frac{n+1}{2}$. If $v_0v_1v_c$ is a face then $v_{n-1}v_0v_{c-1}$, $v_{c-1}v_cv_0$ are faces. Let $v_0v_1v_i$ ($\neq v_0v_1v_c$) be the second face containing v_0v_1 . Then $v_{n-1}v_0v_{i-1}$ is a face and hence $\text{lk}(v_0) = C_6(v_i, v_1, v_c, v_{c-1}, v_{n-1}, v_{i-1})$. Then $v_{i-1}v_iv_0$ is a face and hence $v_0v_1v_{n-i+1}$ is a face. This implies that $n - i + 1 = c$ or i . In either case, we get $i = c$. This is not possible since $v_0v_1v_i \neq v_0v_1v_c$. So, $v_0v_1v_c$ is not a face, where $c = \frac{n+1}{2}$. This proves the claim.

Let $v_0v_1v_k$ be a face containing v_0v_1 . Then, by the claim, $k \in \{3, \dots, \frac{n-1}{2}, \frac{n+3}{2}, \dots, n-2\}$. Now, $v_0v_1v_k \in M$ implies $v_{n-1}v_0v_{k-1}$, $v_{n-k}v_{n-k+1}v_0 \in M$. Then $V(\text{lk}(v_0)) = \{v_{k-1}, v_{n-1}, v_{n-k}, v_{n-k+1}, v_1, v_k\}$ and hence $V(\text{lk}(v_1)) = \{v_k, v_0, v_{n-k+1}, v_{n-k+2}, v_2, v_{k+1}\}$. Thus, $V(\text{lk}(v_0)) \cap V(\text{lk}(v_1)) = \{v_k, v_{n-k+1}\}$ and hence $v_0v_1v_{n-k+1} \in M$. This gives $v_{n-1}v_0v_{n-k} \in M$. Then $\text{lk}(v_0) = C_6(v_{k-1}, v_{n-1}, v_{n-k}, v_{n-k+1}, v_1, v_k)$ and hence $\text{lk}(v_i) = C_6(v_{i+k-1}, v_{n+i-1}, v_{n+i-k}, v_{n+i-k+1}, v_{i+1}, v_{i+k})$ for all i . Now, $M \cong T_{n,1,k-1}$ by the map $\varphi: V(M) \rightarrow \{1, \dots, n\}$ given by $\varphi(v_j) = j$ for $1 \leq i \leq n-1$ and $\varphi(v_0) = n$. \square

LEMMA 3.4. *Let G be a group of order 24×23 . Then G has a unique (and hence normal) subgroup of order 23.*

Proof. Clearly, the number of Sylow 23-subgroups of G is 24 or 1. If possible let there be 24 Sylow 23-subgroups. Let H be a Sylow 23-subgroup. Let $N(H)$ be the normalizer of H in G . Since all the Sylow 23-subgroups are conjugates of H and $|G| = |N(H)| \times |\{\text{conjugates of } H\}|$, $|N(H)| = 23$ and hence $N(H) = H$.

Let A be the set of Sylow 3-subgroups. Then H acts on A by conjugation. Since there is a Sylow 2-subgroup, the number of elements of order 3 is at most 16 and hence $\#(A) < 23$. This implies that the action of H on A is trivial. Let $K \in A$. Then $xKx^{-1} = K$ for all $x \in H$. So, H acts on K by conjugation. Since $|K| = 3$, this action of H on K is trivial. So, $xy = yx$ for all $x \in H$ and $y \in K$. This implies that $K \subseteq N(H)$. This is a contradiction since $N(H) = H$. This proves the lemma. \square

LEMMA 3.5. *Let M be a connected combinatorial 2-manifold. Then the number of flags in M is divisible by the order of $\text{Aut}(M)$.*

Proof. Let $G = \text{Aut}(M)$ and let \mathcal{F} denote the set of flags of M . Then G acts on \mathcal{F} . Let $\sigma \in G$. If there exists a flag $F = (u, uv, uvw)$ such that $\sigma(F) = F$ then $\sigma(v) = v$ and $\sigma(w) = w$. This implies that $\sigma|_{\text{lk}(u)} \equiv \text{Id}$. Since M is connected, this implies that $\sigma \equiv \text{Id}$. Thus, no element of \mathcal{F} is fixed by a non-identity element of G . Therefore, the length of each orbit in \mathcal{F} is same as the order of G . This proves the lemma. \square

LEMMA 3.6. *Let K be a degree-regular triangulation of the torus on $2p$ vertices. If p is prime and ≥ 13 then $\text{Aut}(K)$ has a normal subgroup of order p .*

Proof. Let $G = \text{Aut}(K)$. By Lemma 3.5, $|G|$ is a factor of $2p \times 6 \times 2 = 24p$. By Theorem 1, G acts transitively on $V(K)$. So, the index of the stabilizer of a vertex is $2p$. Thus, $2p$ (and hence p) divides the order of G . Since p is prime, G has an element, say σ , of order p .

Since $|G|$ is a factor of $24p$, by Sylow's theorem, G has a unique Sylow p -subgroup for $p = 13, 17, 19$ or $p > 23$. If $p = 23$ and $|G| < 24p$ then, by Sylow's theorem, G has a unique Sylow p -subgroup. Finally, if $p = 23$ and $|G| = 24 \times 23$ then, by Lemma 3.4, G has a unique Sylow 23-subgroup. Therefore, $H = \langle \sigma \rangle$ is the unique (and hence normal) subgroup of order p in G . \square

Proof of Theorem 2. Let $n \geq 9$ be a composite number. Then either $n = mk$ for some $m, k \geq 3$ or $n = 2p$ for some prime $p \geq 5$. For $m, k \geq 3$, $B_{m,k}$ (defined in Example 4) is an (mk) -vertex degree-regular triangulation of the Klein bottle. If $p \geq 5$ is a prime then $Q_{p,2}$ (defined in Example 6) is an $(2p)$ -vertex degree-regular triangulation of the Klein bottle.

Let $p \geq 13$ be a prime. If possible let there be a p -vertex degree-regular triangulation X of the Klein bottle. Since the torus is an orientable double cover of the Klein bottle, there exists a $(2p)$ -vertex degree-regular triangulation K of the torus and a simplicial covering map $\eta : K \rightarrow X$. Then X is a quotient of K by a subgroup $\langle \tau \rangle$ of $\text{Aut}(K)$, where τ is an automorphism of order 2 without a fixed element (vertex, edge or face). Then $u\tau(u)$ is a non-edge for each $u \in V(K)$. If there exist $u, v \in V(K)$ such that uv and $u\tau(v)$ are edges in K then $\deg_X(\eta(v)) < 6$, a contradiction. So, u and $\tau(u)$ are not adjacent to a common vertex for all $u \in V(K)$.

By Lemma 3.6, there exists a normal subgroup $H \leq \text{Aut}(K)$ of order p . Let $H = \langle \sigma \rangle$. Then $\langle \sigma, \tau \rangle = \langle \sigma \rangle \langle \tau \rangle$ is subgroup of order $2p$. If $\sigma \circ \tau = \tau \circ \sigma$ then $\langle \sigma \rangle$ acts on $K/\langle \tau \rangle$ non-trivially. This implies that X is weakly regular. But, this is not possible by Lemma 3.3. So, $\sigma \circ \tau \neq \tau \circ \sigma$ and hence $\langle \sigma, \tau \rangle \cong D_p$.

Claim 1. No vertex is fixed by σ .

If possible let σ has a fixed vertex. Since K is connected, there is an edge of the form uv such that $\sigma(u) = u$ and $\sigma(v) \neq v$. This implies that $u\sigma^i(v)$ is an edge for all i . Then $\deg(u) \geq p > 6$, a contradiction. This proves the claim.

Claim 2. There exists $w \in V(K)$ and $i \neq 0$ such that $w\sigma^i(w)$ is an edge.

By Claim 1, σ can be written as $\sigma = (u, \sigma(u), \dots, \sigma^{p-1}(u))(v, \sigma(v), \dots, \sigma^{p-1}(v))$ (a permutations on $V(K)$). If $u\sigma^i(u)$ is a non-edge for all i then the link of u is of the form $C_6(\sigma^{i_1}(v), \dots, \sigma^{i_6}(v))$. Then $\sigma^{i_1}(v)\sigma^{i_2}(v)$ is an edge. This implies $v\sigma^{i_2-i_1}(v)$ is an edge. This proves the claim.

By Claim 2, there exists $i \neq 0$ and $w_0 \in V(K)$ such that $w_0\sigma^i(w_0)$ is an edge. Let $\alpha = \sigma^i$. Then $w_0\alpha(w_0)$ is an edge and hence $\alpha^{j-1}(w_0)\alpha^j(w_0)$ is an edge for all j . Since p is prime, $\langle \alpha, \tau \rangle = \langle \sigma, \tau \rangle \cong D_p$. Then $\alpha^j \circ \tau = \tau \circ \alpha^{p-j}$ for all j .

Since p is odd there exists i_0 such that $\tau(\alpha^{i_0}(w_0)) \neq \alpha^j(w_0)$ for any j . Let $u_0 = \alpha^{i_0}(w_0)$, $v_0 = \tau(u_0)$, $u_i = \alpha^i(u_0)$ and $v_i = \alpha^i(v_0)$ for $1 \leq i \leq p-1$. Therefore, $\alpha =$

$(u_0, u_1, \dots, u_{p-1})(v_0, v_1, \dots, v_{p-1})$. Then $\tau(u_i) = \tau(\alpha^i(u_0)) = \alpha^{p-i}(\tau(u_0)) = \alpha^{p-i}(v_0) = v_{p-i}$ and $\tau(v_i) = \tau(\alpha^i(v_0)) = \alpha^{p-i}(\tau(v_0)) = \alpha^{p-i}(u_0) = u_{p-i}$.

Since K is connected, there exists an edge of the form $u_i v_j$ and hence there exists an edge of the form $u_0 v_k$ for some $k \in \{0, \dots, p-1\}$. If k is odd, let $l = \frac{p-k}{2}$. Then $\alpha^l(u_0 v_k)$ is an edge. But, $\alpha^l(u_0 v_k) = u_l v_{k+l} = u_l v_{p-l} = u_l \tau(u_l)$ is a non-edge, a contradiction. If k is even, let $m = \frac{p-k-1}{2}$. Then $\alpha^m(u_0 v_k)$ is an edge. But, $\alpha^m(u_0 v_k) = u_m v_{k+m} = u_m v_{p-m-1} = u_m \tau(u_{m+1})$. This is not possible since $u_m u_{m+1}$ is an edge. This proves that there is no p -vertex degree regular triangulation of the Klein bottle for a prime $p \geq 13$.

If $n = 7, 8$ or 11 then, by Proposition 3, there does not exist any n -vertex degree regular triangulation of the Klein bottle. This completes the proof. \square

Proof of Theorem 3. Since $T_{n,1,k}$ is a weakly regular orientable combinatorial 2-manifold of Euler characteristic 0, Part (a) follows from Part (b) of Lemma 2.1 and Part (b) follows from Parts (b), (c), (i) and (j) of Lemma 2.1. Part (c) follows from Example 6. \square

Proof of Theorem 4. Follows from Lemma 3.3. \square

Proof of Corollary 5. Let M be an n -vertex degree-regular combinatorial 2-manifold of Euler characteristic 0. If n is prime then by Theorem 2, M triangulates the torus and hence, by Theorem 1, M is weakly regular. Then, by Theorem 4, M is isomorphic to $T_{n,1,k}$ for some k . Now, Part (a) follows from Parts (a), (b), (e) and (h) of Lemma 2.1 and Part (b) follows from Parts (a), (b) and (j) of Lemma 2.1. \square

LEMMA 3.7. *Let M be a combinatorial 2-manifold and a_1, \dots, a_5 be five vertices. If the degree of each vertex is 6 then the number of faces in $\text{st}(a_1) \cup \text{st}(a_2) \cup \text{st}(a_3)$ is ≥ 12 and the number of faces in $\text{st}(a_1) \cup \dots \cup \text{st}(a_5)$ is > 12 .*

Proof. Let n be the number of faces in $\text{st}(a_1) \cup \text{st}(a_2) \cup \text{st}(a_3)$. If $a_1 a_2 a_3$ is a face in M then clearly $n = 13$. If $a_1 a_2, a_2 a_3, a_1 a_3$ are edges in M but $a_1 a_2 a_3$ is not a face then $n = 12$. In the other cases, $n \geq 14$.

Let m be the number of faces in $\text{st}(a_1) \cup \dots \cup \text{st}(a_5)$. By above, $m \geq 12$ and $m = 12$ if and only if the induced subcomplex on a set of any three vertices is a K_3 . So, if $m = 12$ then the induced subcomplex of M on $\{a_1, \dots, a_5\}$ is a K_5 and hence the induced subcomplex of $\text{lk}(a_1)$ on $\{a_2, a_3, a_4, a_5\}$ is a null graph on 4 vertices. This is not possible since $\deg(a_1) = 6$. This proves the lemma. \square

LEMMA 3.8. *Let M be an n -vertex connected combinatorial 2-manifold. If the degree of each vertex is 6 and $n > 7$ then for any vertex u there exist faces of the form uab, vab where uv is a non-edge.*

Proof. Let $\text{lk}(u) = C_6(1, \dots, 6)$. Since the degree of each vertex is 6, 123, ..., 456, 561 are not faces. We want to show that there exists $v \notin \{u, 1, \dots, 6\}$ such that $12v, \dots, 56v$ or $16v$ is a face. If not then 124 or 125 is a face. Assume, without loss of generality, that 124 is a face. Then (since $146 \in M \Rightarrow \deg(1) = 4$) the second face containing 16 is 136. Inductively, 256, 145, 346 and 235 are faces. These imply that xy is an edge for $x \neq y \in U := \{u, 1, \dots, 6\}$. Since the degree of each vertex is 6, for $x \in U$ and $z \notin U$, xz is a non-edge and hence (since M has more than 7 vertices) M is not connected. This completes the proof. \square

LEMMA 3.9. *Let M be a connected combinatorial 2-manifold and let U be a set of m vertices of M . If the degree of each vertex in M is 6 and $m < f_0(M)$ then the number of edges in $M[U]$ is at most $3m - 3$.*

Proof. Let V be the vertex-set of M . Let n be the number of edges in $M[U]$ and let k be the number of edges of the form ab , where $a \in U$, $b \in V \setminus U$ (i.e., k is the number of connecting edge between U and $V \setminus U$). Since M is connected and $U \neq V$, $k \neq 0$.

Now, for any connecting edge ab , there exists two faces (with vertices both in U and $V \setminus U$) containing ab . On the other hand, for each such face there exists exactly two connecting edges. This implies $k \geq 3$. If $k \leq 5$ then clearly all faces containing the connecting edges have to be of the form $ab_1b_2, \dots, ab_{k-1}b_k, ab_kb_1$, where $a \in U$ and $b_1, \dots, b_k \in V \setminus U$ or $a \in V \setminus U$ and $b_1, \dots, b_k \in U$. Then $C_5(b_1, \dots, b_k) \subseteq \text{lk}(a)$. This is not possible. So, $k \geq 6$.

Counting two ways the number of pairs of the form (u, e) , where $u \in U$ and e is an edge containing u , we get $6m = n \times 2 + k$ or $2n = 6m - k \leq 6m - 6$. This proves the lemma. \square

LEMMA 3.10. *If M is a 12-vertex degree-regular combinatorial 2-manifolds of Euler characteristic 0 then M is isomorphic to $T_{12,1,2}$, $T_{12,1,3}$, $T_{12,1,4}$, $T_{6,2,2}$, $B_{3,4}$, $B_{4,3}$ or $K_{3,4}$.*

Proof. Let M be a 12-vertex degree regular combinatorial 2-manifold of Euler characteristic 0. Let the vertex set V of M be $\{0, \dots, 9, u, v\}$. Let $\varphi: V \rightarrow \{1, \dots, 12\}$ be given by $\varphi(i) = i$ for $1 \leq i \leq 9$, $\varphi(0) = 10$, $\varphi(u) = 11$ and $\varphi(v) = 12$.

Since $\chi(M) = 0$, the degree of each vertex is 6. Assume, without loss of generality, that $\text{lk}(0) = C_6(1, \dots, 6)$. Since the degree of each vertex is 6, $123, \dots, 456, 561 \notin M$. Since each component contains at least 7 vertices, M is connected. So, by Lemma 3.8, we may assume that 127 is a face. Then $\text{lk}(1)$ has the form $C_6(7, 2, 0, 6, x, y)$, for some $x, y \in V$. It is easy to see that $(x, y) = (3, 4), (3, 5), (3, 8), (4, 3), (4, 5), (4, 8), (8, 4), (8, 9), (8, 3), (8, 5)$. The cases $(x, y) = (8, 3)$ and $(8, 5)$ are isomorphic to the case $(x, y) = (4, 8)$ by the map $(0, 1)(2, 6)(3, 4, 8)(5, 7)$ and $(0, 1)(3, 7)(4, 8, 5)$ respectively. So, we need not consider the last two cases.

Claim. $(x, y) = (3, 4), (3, 8), (4, 3), (4, 8), (8, 4)$ or $(8, 9)$.

If $(x, y) = (3, 5)$, then 045, 056, 135, 157 are faces and hence $\text{lk}(5) = C_6(4, 0, 6, 3, 1, 7)$. This implies that $C_3(1, 5, 6) \subseteq \text{lk}(3)$. This is not possible.

If $(x, y) = (4, 5)$ then $\text{lk}(4) = C_6(6, 1, 5, 0, 3, z)$, where $z = 7, 8, 9, u$ or v . If $z = 7$ then, as in the previous case, we get a contradiction. So, we may assume that $\text{lk}(4) = C_6(6, 1, 5, 0, 3, 8)$ and hence $\text{lk}(6) = C_6(8, 4, 1, 0, 5, w)$, $\text{lk}(5) = C_6(7, 1, 4, 0, 6, w)$ for some $w \in V$. It is easy to see that $w = 9, u$ or v . In any case, we get 15 faces not containing any from $\{9, u, v\} \setminus \{w\}$. This is not possible since M has 24 faces. This proves the claim.

Case 1. $(x, y) = (3, 4)$, i.e., $\text{lk}(1) = C_6(7, 2, 0, 6, 3, 4)$. Now, $\text{lk}(3) = C_6(6, 1, 4, 0, 2, z)$ for some $z \in V$. If $z = 5$ then $C_4(5, 3, 1, 0) \subseteq \text{lk}(6)$. If $z = 7$ then $C_4(7, 3, 0, 1) \subseteq \text{lk}(2)$. This implies that $z \in \{8, 9, u, v\}$. Assume, without loss of generality, that $\text{lk}(3) = C_6(6, 1, 4, 0, 2, 8)$. Now, $\text{lk}(2) = C_6(8, 3, 0, 1, 7, w)$, for some $w \in V$. If $w \notin \{9, u, v\}$ then we get 14 faces not containing any of 9, u , v . This is not possible by Lemma 3.7. So, assume without loss of generality, that $z = 9$, i.e., $\text{lk}(2) = C_6(8, 3, 0, 1, 7, 9)$.

Completing successively, we get $\text{lk}(6) = C_6(8, 3, 1, 0, 5, u)$, $\text{lk}(8) = C_6(u, 6, 3, 2, 9, v)$, $\text{lk}(4) = C_6(7, 1, 3, 0, 5, v)$, $\text{lk}(5) = C_6(v, 4, 0, 6, u, 9)$, $\text{lk}(9) = C_6(7, u, 5, v, 8, 2)$, $\text{lk}(u) = C_6(7, 9, 5, 6, 8, v)$. Here $M \cong B_{3,4}$ by the map $\varphi_{34} \circ (0, 9, 3, 4, 7, u, 2, v)(1, 8)(5, 6)$, where $\varphi_{34}: V \rightarrow V(B_{3,4})$ is given by $\varphi_{34}(i) = v_{1i}$, $\varphi_{34}(3+i) = v_{2i}$, $\varphi_{34}(6+i) = v_{3i}$, for $1 \leq i \leq 3$, $\varphi_{34}(0) = v_{41}$, $\varphi_{34}(u) = v_{42}$, $\varphi_{34}(v) = v_{43}$.

Case 2. $(x, y) = (3, 8)$, i.e., $\text{lk}(1) = C_6(7, 2, 0, 6, 3, 8)$. Now, 023, 034, 136 and 138 are faces in M . So, $\text{lk}(3) = C_6(2, 0, 4, 8, 1, 6)$ or $C_6(2, 0, 4, 6, 1, 8)$. In the first case, $\text{lk}(2) = C_6(6, 3, 0, 1, 7, z)$ for some $z \in V$. As in Case 1, z is a new vertex, say, 9. Then $\text{lk}(2) = C_6(6, 3, 0, 1, 7, 9)$ and $\text{lk}(6) = C_6(9, 2, 3, 1, 0, 5)$. This gives 15 faces not containing u or v . This is not possible. Thus, $\text{lk}(3) = C_6(2, 0, 4, 6, 1, 8)$. Now, $\text{lk}(6) = C_6(4, 3, 1, 0, 5, w)$ for some $w \in V$. If $w = 2, 7$ or 8 , then we get 14 faces not containing any of $9, u, v$. This is not possible by Lemma 3.7. So, assume without loss, that $\text{lk}(6) = C_6(4, 3, 1, 0, 5, 9)$.

Completing successively, we get $\text{lk}(4) = C_6(9, 6, 3, 0, 5, u)$, $\text{lk}(5) = C_6(4, 0, 6, 9, v, u)$, $\text{lk}(9) = C_6(4, 6, 5, v, 7, u)$, $\text{lk}(7) = C_6(9, u, 8, 1, 2, v)$, $\text{lk}(2) = C_6(7, 1, 0, 3, 8, v)$, $\text{lk}(8) = C_6(2, 3, 1, 7, u, v)$. Here $M \cong T_{12,1,2}$ by the map $\varphi \circ (0, 4, 6, 5, 7, u, 9, 8, v)(1, 2)$.

Case 3. $(x, y) = (4, 3)$, i.e., $\text{lk}(1) = C_6(7, 2, 0, 6, 4, 3)$. Now, $\text{lk}(4) = C_6(6, 1, 3, 0, 5, z)$ for some $z \in V$. If $z = 2$, then $\text{lk}(2)$ has 7 vertices. If $z = 7$, then $\text{lk}(4) = C_6(6, 1, 3, 0, 5, 7)$ and hence 127, 137, 457, 467 are faces in M . This implies that $\text{lk}(7) = C_6(2, 1, 3, 5, 4, 6)$ or $C_6(2, 1, 3, 6, 4, 5)$. In either cases we get 14 faces not containing any of $8, 9, u$ or v . This is not possible by Lemma 3.7. This implies that $z = 8, 9, u$ or v . Assume, without loss, that $z = 8$. This case is now isomorphic to Case 2 by the map $(0, 3, 6, 2, 4, 1)(5, 8, 7)(9, v)$.

Case 4. $(x, y) = (4, 8)$, i.e., $\text{lk}(1) = C_6(6, 0, 2, 7, 8, 4)$. This gives $\text{lk}(4) = C_6(8, 1, 6, 3, 0, 5)$.

Now, $\text{lk}(6) = C_6(3, 4, 1, 0, 5, z)$ for some $z \in V$. By using Lemma 3.7, $z = 9, u$ or v . So, assume that $\text{lk}(6) = C_6(3, 4, 1, 0, 5, 9)$. This implies that $\text{lk}(5) = C_6(9, 6, 0, 4, 8, u)$. This case is now isomorphic to Case 1 by the map $(1, 6)(2, 5)(3, 4)(9, 7, u)$.

Case 5. $(x, y) = (8, 4)$, i.e., $\text{lk}(1) = C_6(6, 0, 2, 7, 4, 8)$. Now, 034, 045, 147 and 148 are faces in M . So, $\text{lk}(4) = C_6(3, 0, 5, 7, 1, 8)$ or $C_6(3, 0, 5, 8, 1, 7)$.

Subcase 5.1. $\text{lk}(4) = C_6(3, 0, 5, 7, 1, 8)$. Then (by using Lemma 3.7) $\text{lk}(7) = C_6(5, 4, 1, 2, z, w)$, where $z, w \in \{9, u, v\}$. So, assume without loss, that $\text{lk}(7) = C_6(5, 4, 1, 2, 9, u)$. Completing successively, we get $\text{lk}(2) = C_6(9, 7, 1, 0, 3, v)$, $\text{lk}(3) = C_6(2, 0, 4, 8, u, v)$, $\text{lk}(u) = C_6(8, 9, 7, 5, v, 3)$, $\text{lk}(8) = C_6(9, u, 3, 4, 1, 6)$, $\text{lk}(5) = C_6(7, 4, 0, 6, v, u)$, $\text{lk}(6) = C_6(5, 0, 1, 8, 9, v)$. Here $M \cong K_{3,4}$ by the map $\psi_{34} \circ (0, 8)(1, 9)(2, v, 3, u)(4, 7)$, where ψ_{34} is same as φ_{34} (of Case 1) on the vertex-set.

Subcase 5.2. $\text{lk}(4) = C_6(3, 0, 5, 8, 1, 7)$. Then $\text{lk}(7) = C_6(3, 4, 1, 2, z, w)$, for some $z, w \in V$. As in the previous case, $z, w \in \{9, u, v\}$. So, assume without loss, that $\text{lk}(7) = C_6(3, 4, 1, 2, 9, u)$. Then $\text{lk}(2) = C_6(9, 7, 1, 0, 3, a)$ for some $a \in V$. It is easy to see that $a = 8$ or v . If $a = 8$ then, considering $\text{lk}(8)$, we get 19 faces not containing v . This is not possible since $f_2(M) = 24$. So, $\text{lk}(2) = C_6(9, 7, 1, 0, 3, v)$ and hence $\text{lk}(3) = C_6(2, 0, 4, 7, u, v)$. Then, $\text{lk}(8) = C_6(5, 4, 1, 6, b, c)$, where $b, c \in \{9, u, v\}$. Since the set of known faces is invariant under $(1, 4)(2, 3)(5, 6)(9, u)$, we may assume that $(b, c) = (9, u), (u, 9), (v, u)$ or $(v, 9)$.

Subcase 5.2.1. $\text{lk}(8) = C_6(5, 4, 1, 6, 9, u)$. Completing successively, we get $\text{lk}(9) = C_6(2, 7, u, 8, 6, v)$, $\text{lk}(6) = C_6(9, 8, 1, 0, 5, v)$, $\text{lk}(5) = C_6(6, 0, 4, 8, u, v)$. Here $M \cong T_{12,1,4}$ by the map $\varphi \circ (0, 2, 4, u, 9, 6)(1, v, 8)(3, 5, 7)$.

Subcase 5.2.2. $\text{lk}(8) = C_6(5, 4, 1, 6, u, 9)$. Completing successively, we get $\text{lk}(u) = C_6(6, 8, 9, 7, 3, v)$, $\text{lk}(9) = C_6(5, 8, u, 7, 2, v)$, $\text{lk}(v) = C_6(6, u, 3, 2, 9, 5)$. Here $M \cong B_{4,3}$ by the map $\varphi_{43} \circ (0, 9, u, 7, 6, 8, v, 3, 2)(1, 5, 4)$, where $\varphi_{43}: V \rightarrow V(B_{4,3})$ is given by $\varphi_{43}(i) = v_{1i}$, $\varphi_{43}(4+i) = v_{2i}$, for $1 \leq i \leq 4$, $\varphi_{43}(9) = v_{31}$, $\varphi_{43}(0) = v_{32}$, $\varphi_{43}(u) = v_{33}$, $\varphi_{43}(v) = v_{34}$.

Subcase 5.2.3. $\text{lk}(8) = C_6(5, 4, 1, 6, v, 9)$. Completing successively, we get $\text{lk}(v) = C_6(3, 2, 9, 8, 6, u)$, $\text{lk}(9) = C_6(8, v, 2, 7, u, 5)$, $\text{lk}(u) = C_6(6, 5, 9, 7, 3, v)$. Here $M \cong T_{12,1,3}$ by the map $\varphi \circ (1, 0, 8, 6)(2, 5, u, 4, 3, v, 9)$.

Subcase 5.2.4. $\text{lk}(8) = C_6(5, 4, 1, 6, v, u)$. Completing successively, we get $\text{lk}(v) =$

$C_6(3, 2, 9, 6, 8, u)$, $\text{lk}(u) = C_6(3, 7, 9, 5, 8, v)$, $\text{lk}(9) = C_6(6, 5, u, 7, 2, v)$. Here $M \cong B_{4,3}$ by the map $\varphi_{43} \circ (1, 2, 9, 4, 6, 3, 5, u, v, 8, 7)$, where φ_{43} is as in Subcase 5.2.2.

Case 6. $(x, y) = (8, 9)$, i.e., $\text{lk}(1) = C_6(6, 0, 2, 7, 9, 8)$. Now, $\text{lk}(2) = C_6(7, 1, 0, 3, z, w)$ for some $z, w \in V$. It is easy to see that $(z, w) = (5, 6), (6, 8), (5, 8), (5, u), (u, 8), (6, 5), (8, 6), (8, u), (u, 5), (9, u), (u, 4), (u, v)$. If $(z, w) = (5, 8)$, i.e., $\text{lk}(2) = C_6(7, 1, 0, 3, 5, 8)$ then 045, 056, 235, 258 are faces in M . This implies that $\text{lk}(5) = C_6(6, 0, 4, 8, 2, 3)$. Then $\deg(8) \geq 7$. Since the set of known faces are invariant under $(0, 1)(3, 7)(4, 9)(5, 8)$, we may assume that $(z, w) = (5, 6), (5, u), (6, 5), (u, 5), (9, u)$ or (u, v) .

Subcase 6.1. $\text{lk}(2) = C_6(7, 1, 0, 3, 5, 6)$. Now, it is easy to see that $\text{lk}(6) = C_6(7, 2, 5, 0, 1, 8)$. Now, completing successively, we get $\text{lk}(7) = C_6(8, 6, 2, 1, 9, u)$, $\text{lk}(8) = C_6(u, 7, 6, 1, 9, v)$, $\text{lk}(9) = C_6(8, 1, 7, u, 4, v)$, $\text{lk}(4) = C_6(3, 0, 5, v, 9, u)$, $\text{lk}(5) = C_6(4, 0, 6, 2, 3, v)$, $\text{lk}(3) = C_6(5, 2, 0, 4, u, v)$. Here $M \cong T_{12,1,2}$ by the map $\varphi \circ (0, 9, 3, u, 2, 8, 4, v, 1, 6, 7, 5)$.

Subcase 6.2. $\text{lk}(2) = C_6(7, 1, 0, 3, 5, u)$. Now, it is easy to see that $\text{lk}(5) = C_6(4, 0, 6, 3, 2, u)$. Now, completing successively, we get $\text{lk}(3) = C_6(2, 0, 4, v, 6, 5)$, $\text{lk}(4) = C_6(3, 0, 5, u, 9, v)$, $\text{lk}(9) = C_6(8, 1, 7, v, 4, u)$, $\text{lk}(7) = C_6(9, 1, 2, u, 8, v)$, $\text{lk}(8) = C_6(1, 6, v, 7, u, 9)$, $\text{lk}(u) = C_6(2, 5, 4, 9, 8, 7)$. Here $M \cong B_{3,4}$ by the map $\varphi_{34} \circ (0, 9, 3, 4, v, 1, 6, 5, 8, 2, 7)$, where φ_{34} is as in Case 1.

Subcase 6.3. $\text{lk}(2) = C_6(7, 1, 0, 3, 6, 5)$. Completing successively, we get $\text{lk}(6) = C_6(5, 0, 1, 8, 3, 2)$, $\text{lk}(3) = C_6(2, 0, 4, u, 8, 6)$, $\text{lk}(5) = C_6(6, 0, 4, v, 7, 2)$, $\text{lk}(8) = C_6(3, 6, 1, 9, v, u)$, $\text{lk}(7) = C_6(2, 1, 9, u, v, 5)$, $\text{lk}(4) = C_6(5, 0, 3, u, 9, v)$, $\text{lk}(u) = C_6(4, 3, 8, v, 7, 9)$. Here $M \cong B_{3,4}$ by the map $\varphi_{34} \circ (0, 9, 3, 5, 7, u, 2, 8, 1, v)(4, 6)$, where φ_{34} is as in Case 1.

Subcase 6.4. $\text{lk}(2) = C_6(7, 1, 0, 3, 9, u)$. Completing successively, we get $\text{lk}(9) = C_6(8, 1, 7, 3, 2, u)$, $\text{lk}(3) = C_6(2, 0, 4, v, 7, 9)$, $\text{lk}(7) = C_6(3, 9, 1, 2, u, v)$, $\text{lk}(u) = C_6(7, 2, 9, 8, 5, v)$, $\text{lk}(8) = C_6(5, u, 9, 1, 6, 4)$, $\text{lk}(5) = C_6(8, 4, 0, 6, v, u)$, $\text{lk}(6) = C_6(5, 0, 1, 8, 4, v)$. Here $M \cong B_{3,4}$ by the map $\varphi_{34} \circ (0, 7, 2)(1, u)(3, 6, 8, v, 5, 4, 9)$, where φ_{34} is as in Case 1.

Subcase 6.5. $\text{lk}(2) = C_6(7, 1, 0, 3, u, 5)$. Now, it is easy to see that $\text{lk}(5) = C_6(4, 0, 6, u, 2, 7)$ or $C_6(4, 0, 6, 7, 2, u)$. The first case is isomorphic to Subcase 5.1 by the map $(0, 1, 7, 5)(2, 4, 6)(3, 8, u)$. The second case is isomorphic to Subcase 5.2 by the map $(0, 4, 5)(1, 7, 2)(3, 8, u, 6)$.

Subcase 6.6. $\text{lk}(2) = C_6(7, 1, 0, 3, u, v)$. Now, it is easy to see that $\text{lk}(7) = C_7(v, 2, 1, 9, a, b)$, where $(a, b) = (3, 4), (4, 3), (4, 5), (4, 8), (5, 4), (5, 6), (5, 8), (6, 5), (6, 8), (u, 3), (u, 4), (u, 5), (u, 8)$. Since the set of known faces is invariant under the map $(1, 2)(3, 6)(4, 5)(8, u)(9, v)$, we may assume that $(a, b) = (3, 4), (4, 3), (4, 5), (4, 8), (5, 4), (5, 8), (6, 8), (u, 8)$.

Claim. $(a, b) = (3, 4)$ or $(5, 4)$.

If $(a, b) = (4, 3)$ then, considering $\text{lk}(3)$, we get $C_4(u, 3, 7, 2) \subseteq \text{lk}(v)$. If $(a, b) = (4, 5)$ then, $\text{lk}(5)$ can not be a 6-cycle. If $(a, b) = (4, 8)$ then, considering $\text{lk}(9)$, we see that $0, 1, 4, 7, 8, 9 \notin \text{lk}(u)$. This is not possible. If $(a, b) = (5, 8)$ then, considering $\text{lk}(8)$, we get $C_4(5, 8, 1, 0) \subseteq \text{lk}(6)$. If $(a, b) = (6, 8)$ then, considering the links of 6, 9 and 8 successively, we get $C_4(u, 8, 7, 2) \subseteq \text{lk}(v)$. If $(a, b) = (u, 8)$ then, considering $\text{lk}(u)$, we get 7 vertices in $\text{lk}(8)$. These prove the claim.

Subcase 6.6.1. $\text{lk}(7) = C_6(v, 2, 1, 9, 3, 4)$. Completing successively, we get $\text{lk}(3) = C_6(0, 2, u, 9, 7, 4)$, $\text{lk}(9) = C_6(1, 7, 3, u, 5, 8)$, $\text{lk}(4) = C_6(0, 3, 7, v, 8, 5)$, $\text{lk}(5) = C_6(0, 4, 8, 9, u, 6)$, $\text{lk}(u) = C_6(2, 3, 9, 5, 6, v)$, $\text{lk}(6) = C_6(0, 1, 8, v, u, 5)$. Now, $M \cong K_{3,4}$ by the map $\psi_{34} \circ (0, 4, 5, 7, 3, 2, 1, v, 6, 9, u)$, where ψ_{34} is as in Subcase 5.1.

Subcase 6.6.2. $\text{lk}(7) = C_6(v, 2, 1, 9, 5, 4)$. Now, it is easy to see that $\text{lk}(4) = C_6(3, 0, 5, 7, v, 8)$. Now, $0, 1, 4, 7, 8, 9 \notin \text{lk}(u)$. So, $5 \in \text{lk}(u)$ and hence $\text{lk}(5) = C_6(6, 0, 4, 7, 9, u)$. Then $\text{lk}(3) = C_6(8, 4, 0, 2, u, 6)$ or $C_6(8, 4, 0, 2, u, 9)$.

Subcase 6.6.2.1. $\text{lk}(3) = C_6(8, 4, 0, 2, u, 6)$. Completing successively, we get $\text{lk}(6) = C_6(0, 1, 8, 3, u, 5)$, $\text{lk}(8) = C_6(1, 6, 3, 4, v, 9)$, $\text{lk}(9) = C_6(1, 7, 5, u, v, 8)$. Now, $M \cong K_{3,4}$ by the map $\psi_{34} \circ (0, 8, 4, v, 1, 5, u)(2, 6, 7, 3, 9)$, where ψ_{34} is as in Subcase 5.1.

Subcase 6.6.2.2. $\text{lk}(3) = C_6(8, 4, 0, 2, u, 9)$. Completing successively, we get $\text{lk}(9) = C_6(1, 7, 5, u, 3, 8)$, $\text{lk}(8) = C_6(1, 6, v, 4, 3, 9)$, $\text{lk}(6) = C_6(0, 1, 8, v, u, 5)$. Now, $M \cong T_{6,2,2}$ by the map $\psi \circ (0, 1, u, 7, 4, 3, 8)(2, 6, v, 5)$, where $\psi: V \rightarrow V(T_{6,2,2})$ is given by $\psi(i) = u_i$, for $1 \leq i \leq 6$, $\psi(5+i) = v_i$, for $2 \leq i \leq 4$, $\psi(0) = v_5$, $\psi(u) = v_6$ and $\psi(v) = v_7$. \square

LEMMA 3.11. *If M is a 14-vertex degree-regular combinatorial 2-manifolds of Euler characteristic 0 then M is isomorphic to $T_{14,1,2}$, $T_{14,1,3}$ or $Q_{7,2}$.*

Proof. Let M be a 14-vertex degree regular combinatorial 2-manifold of Euler characteristic 0. Let the vertex set V be $\{0, 1, \dots, 9, u, v, w, z\}$. Let $\varphi: V \rightarrow \{1, \dots, 14\}$ be given by $\varphi(i) = i$, for $1 \leq i \leq 9$, $\varphi(0) = 10$, $\varphi(u) = 11$, $\varphi(v) = 12$, $\varphi(w) = 13$ and $\varphi(z) = 14$.

Since $\chi(M) = 0$, the degree of each vertex is 6. Assume without loss that $\text{lk}(0) = C_6(1, 2, 3, 4, 5, 6)$. By Lemma 3.8, $\text{lk}(1) = C_6(6, 0, 2, 7, x, y)$, for some $x, y \in V$. It is easy to see that $(x, y) = (3, 4), (3, 8), (4, 3), (4, 8), (5, 3), (5, 4), (5, 8), (8, 3), (8, 4), (8, 9)$. The case $(x, y) = (3, 8)$ is isomorphic to the case $(x, y) = (5, 8)$ by the map $(2, 6)(3, 5)(7, 8)$ and to the case $(x, y) = (8, 4)$ by the map $(0, 1)(2, 6)(3, 4, 8)(5, 7)$. Hence we may assume that $(x, y) = (3, 4), (3, 8), (4, 3), (4, 8), (5, 3), (5, 4), (8, 3), (8, 9)$.

Claim. $(x, y) = (3, 4), (8, 3)$ or $(8, 9)$

If $(x, y) = (4, 3)$ then, considering the links of $3, 6, 2, 4, 7, 8, u$ successively we get $C_3(v, u, 7) \subseteq \text{lk}(w)$. So, $(x, y) \neq (4, 3)$. If $(x, y) = (5, 3)$ then, $\text{lk}(5) = C_6(1, 3, 6, 0, 4, 7)$. But then $C_4(0, 1, 3, 5) \subseteq \text{lk}(6)$. So, $(x, y) \neq (5, 3)$. Similarly, $(x, y) \neq (3, 8), (4, 8)$ or $(5, 4)$. This proves the claim.

Case 1. $(x, y) = (3, 4)$, i.e., $\text{lk}(1) = C_6(6, 0, 2, 7, 3, 4)$. Now, it is easy to see that $\text{lk}(3) = C_6(7, 1, 4, 0, 2, 5)$ or $C_6(7, 1, 4, 0, 2, 8)$. In the first case, $\text{lk}(5) = C_6(4, 0, 6, 2, 3, 7)$ or $C_6(4, 0, 6, 7, 3, 2)$. In both these cases we have 23 edges in $M[\{0, \dots, 7\}]$. This is not possible by Lemma 3.9. Thus, $\text{lk}(3) = C_6(7, 1, 4, 0, 2, 8)$. Now, $\text{lk}(2) = C_6(7, 1, 0, 3, 8, b)$ for some $b \in V$. It is easy to see that $b = 5$ or 9 . If $b = 5$ then, considering the links of 4 and 5, we get ≥ 17 faces not containing any of $9, u, v, w, z$. This is not possible by Lemma 3.7. Thus $\text{lk}(2) = C_6(7, 1, 0, 3, 8, 9)$. Again, by using Lemma 3.7, we successively get $\text{lk}(7) = C_6(8, 3, 1, 2, 9, u)$, $\text{lk}(8) = C_6(9, 2, 3, 7, u, v)$, $\text{lk}(9) = C_6(u, 7, 2, 8, v, w)$, $\text{lk}(u) = C_6(v, 8, 7, 9, w, z)$, $\text{lk}(v) = C_6(w, 9, 8, u, z, 5)$. Then $045, 056, 5vw$ and $5vz$ are faces. So, $\text{lk}(5) = C_6(6, 0, 4, w, v, z)$ or $C_6(6, 0, 4, z, v, w)$. In the first case, considering the links of 4 and 6, we get $C_5(w, 6, 5, v, u) \subseteq \text{lk}(z)$. Thus $\text{lk}(5) = C_6(6, 0, 4, z, v, w)$.

Now, completing successively, we get $\text{lk}(w) = C_6(9, v, 5, 6, z, u)$ and $\text{lk}(6) = C_6(1, 0, 5, w, z, 4)$. Here M is isomorphic to $T_{14,1,2}$ by the map $\varphi \circ (0, 7, 3, 5)(1, 6, 9)(2, 4, 8)(u, z)(v, w)$.

Case 2. $(x, y) = (8, 3)$. Since 023, 034, 136 and 138 are faces, $\text{lk}(3) = C_6(2, 0, 4, 8, 1, 6)$ or $C_6(2, 0, 4, 6, 1, 8)$. In the first case, considering the links of 6, 2, 9, 7, 8 and 4 successively, we get 7 vertices in $\text{lk}(u)$. Thus, $\text{lk}(3) = C_6(2, 0, 4, 6, 1, 8)$.

Now, completing successively, we get $\text{lk}(6) = C_6(5, 0, 1, 3, 4, 9)$, $\text{lk}(4) = C_6(5, 0, 3, 6, 9, u)$, $\text{lk}(5) = C_6(u, 4, 0, 6, 9, v)$, $\text{lk}(9) = C_6(u, 4, 6, 5, v, w)$, $\text{lk}(2) = C_6(8, 3, 0, 1, 7, z)$, $\text{lk}(8) = C_6(7, 1, 3, 2, z, w)$, $\text{lk}(u) = C_6(4, 5, v, z, w, 9)$, $\text{lk}(z) = C_6(2, 7, v, u, w, 8)$ and $\text{lk}(7) = C_6(1, 2, z, v, w, 8)$. Here $M \cong T_{14,1,2}$ by the map $\varphi \circ (0, 6, 7, 1, 4, 8, 2, 3, 5, 9)$.

Case 3. $(x, y) = (8, 9)$, i.e., $\text{lk}(1) = C_6(6, 0, 2, 7, 8, 9)$. Now, $\text{lk}(6) = C_6(9, 1, 0, 5, a, b)$, for some $a, b \in V$. It is easy to see that $(a, b) = (2, 3), (2, 7), (3, 2), (3, 4), (3, 7), (3, u), (7, 2), (7, 3), (7, 4), (7, u), (8, 3), (8, 4), (8, 7), (8, u), (u, 3), (u, 4), (u, 7), (u, v)$. Since the set of

known faces is invariant under the map $(0, 1)(3, 7)(4, 8)(5, 9)$, we may assume that $(a, b) = (2, 3), (2, 7), (3, 4), (3, 7), (3, u), (7, 3), (7, 4), (7, u), (8, 4), (8, u)$ or (u, v) .

By the similar arguments as in the previous claim one gets $(a, b) = (2, 7)$ or (u, v) .

Subcase 3.1. $\text{lk}(6) = C_6(9, 1, 0, 5, 2, 7)$. Completing successively, we get $\text{lk}(2) = C_6(1, 0, 3, 5, 6, 7)$, $\text{lk}(5) = C_6(4, 0, 6, 2, 3, u)$, $\text{lk}(3) = C_6(2, 0, 4, v, u, 5)$, $\text{lk}(4) = C_6(3, 0, 5, u, w, v)$, $\text{lk}(u) = C_6(3, 5, 4, w, z, v)$, $\text{lk}(7) = C_6(1, 2, 6, 9, z, 8)$, $\text{lk}(9) = C_6(1, 6, 7, z, w, 8)$, $\text{lk}(z) = C_6(7, 8, v, u, w, 9)$ and $\text{lk}(v) = C_6(3, 4, w, 8, z, u)$. Here M is isomorphic to $T_{14,1,2}$ by the map $\varphi \circ (0, 4, 1, 7, 8)(2, 5, 3)(u, z)(v, w)$.

Subcase 3.2. $\text{lk}(6) = C_6(9, 1, 0, 5, u, v)$. Now, $\text{lk}(5) = C_6(u, 6, 0, 4, c, d)$, for some $c, d \in V$. It is easy to see that $(c, d) = (2, 3), (2, 7), (7, 2), (7, 3), (7, 8), (7, w), (8, 3), (8, 7), (8, 9), (8, w), (9, 8), (v, 3), (v, 7), (v, 8), (v, 9), (v, w), (w, 3), (w, 7), (w, 8), (w, z)$. Since the set of known faces is invariant under the map $(0, 6)(2, 9)(3, v)(4, u)(7, 8)$, we may assume that $(c, d) = (2, 3), (2, 7), (7, 2), (7, 3), (7, 8), (7, w), (8, 3), (8, 7), (8, w), (v, 3), (v, w)$ or (w, z) .

Claim. $(c, d) = (7, 8), (8, w)$ or (w, z) .

If $(c, d) = (2, 3)$ then, $\text{lk}(5) = C_6(u, 6, 0, 4, 2, 3)$. Considering the links of 2, 4, 3, u , w successively, we get $C_5(8, w, 4, 2, 1) \subseteq \text{lk}(7)$. If $(c, d) = (2, 7)$ then, considering $\text{lk}(2)$ we get $C_4(3, 2, 5, 0) \subseteq \text{lk}(4)$. If $(c, d) = (7, 3)$ then, considering $\text{lk}(3)$, we get $C_4(7, 3, 0, 1) \subseteq \text{lk}(2)$. If $(c, d) = (v, 3)$ then, considering $\text{lk}(3)$, we get 7 vertices in $\text{lk}(v)$. So, $(c, d) \neq (2, 3), (2, 7), (7, 3)$ or $(v, 3)$. Similarly, $(c, d) \neq (7, 2), (7, w), (8, 3), (8, 7)$ or (v, w) . This proves the claim.

Subcase 3.2.1. $\text{lk}(5) = C_6(u, 6, 0, 4, 7, 8)$. Now, $\text{lk}(8) = C_6(u, 5, 7, 1, 9, x)$, for some $x \in V$. It is easy to check that $x = 3, w$. By using Lemma 3.7, we get $x \neq 3$. So, $\text{lk}(8) = C_6(u, 5, 7, 1, 9, w)$. This implies that $\text{lk}(9) = C_6(w, 8, 1, 6, v, y)$, for some $y \in V$. It is easy to see that $y = 3$ or z .

Subcase 3.2.1.1. $\text{lk}(9) = C_6(w, 8, 1, 6, v, 3)$. This implies that $\text{lk}(u) = C_6(w, 8, 5, 6, v, z)$. Again, by using Lemma 3.7, we get $\text{lk}(7) = C_6(4, 5, 8, 1, 2, z)$. Then $\text{lk}(4) = C_6(3, 0, 5, 7, z, a)$, for some $a \in V$. Considering $\text{lk}(3)$, we get $a = v$ or w .

A. $\text{lk}(4) = C_6(3, 0, 5, 7, z, v)$. Completing successively we get $\text{lk}(z) = C_6(2, 7, 4, v, u, w)$, $\text{lk}(2) = C_6(1, 0, 3, w, z, 7)$ and $\text{lk}(3) = C_6(2, 0, 4, v, 9, w)$. Here $M \cong T_{14,1,3}$ by the map $\varphi \circ (0, 1, 4, v, w, 6, z, 9, 3, 2, 5, u)(7, 8)$.

B. $\text{lk}(4) = C_6(3, 0, 5, 7, z, w)$. Completing successively, we get $\text{lk}(3) = C_6(2, 0, 4, w, 9, v)$, $\text{lk}(2) = C_6(1, 0, 3, v, z, 7)$ and $\text{lk}(v) = C_6(2, 3, 9, 6, u, z)$. Now, $M \cong Q_{7,2}$ by the map $\varphi \circ (0, 1, 9, u, v, 5, 2, 7, 8)(3, w, 4, z, 6)$.

Subcase 3.2.1.2. $\text{lk}(9) = C_6(w, 8, 1, 6, v, z)$. Then it follows that $\text{lk}(u) = C_6(v, 6, 5, 8, w, 3)$. This case is now isomorphic to the Subcase 3.2.1.1 by the map $(1, 5)(2, 4)(u, 9)$.

Subcase 3.2.2. $\text{lk}(5) = C_6(u, 6, 0, 4, 8, w)$. Since, 178, 189, 458 and 58w are faces, $\text{lk}(8) = C_6(4, 5, w, 9, 1, 7)$ or $C_6(4, 5, w, 7, 1, 9)$.

Subcase 3.2.2.1. $\text{lk}(8) = C_6(4, 5, w, 9, 1, 7)$. Then $\text{lk}(4) = C_6(3, 0, 5, 8, 7, x)$, for some $x \in V$. It is easy to see that $x = v, z$. If $x = v$ then, considering the links of 4, 7, v successively, we obtain 29 faces in M , which is not possible. Thus $\text{lk}(4) = C_6(3, 0, 5, 8, 7, z)$.

Completing successively, we get $\text{lk}(7) = C_6(2, 1, 8, 4, z, v)$, $\text{lk}(2) = C_6(3, 0, 1, 7, v, u)$, $\text{lk}(v) = C_6(2, u, 6, 9, z, 7)$, $\text{lk}(9) = C_6(1, 6, v, z, w, 8)$, $\text{lk}(z) = C_6(3, 4, 7, v, 9, w)$ and $\text{lk}(w) = C_6(3, u, 5, 8, 9, z)$. Here $M \cong T_{14,1,3}$ by the map $\varphi \circ (0, 6, 2, 9, 1, 5, 3)(4, 7, 8)(u, w, z)$.

Subcase 3.2.2.2. $\text{lk}(8) = C_6(4, 5, w, 7, 1, 9)$. Then $\text{lk}(4) = C_6(3, 0, 5, 8, 9, z)$ and $\text{lk}(9) = C_6(1, 6, v, z, 4, 8)$. Now, $\text{lk}(w) = C_6(u, 5, 8, 7, a, b)$, for some $a, b \in V$. It is easy to check that $(a, b) = (3, 2), (3, z), (v, z)$ or $(z, 3)$. The set of known faces is invariant under the map $(0, 9)(1, 6)(2, v)(3, z)(5, 8)(7, u)$. So, we may assume that $(a, b) = (3, 2), (3, z)$ or $(z, 3)$. If

$(a, b) = (3, 2)$ then, considering the links of 2 and 3, we get 7 vertices in $\text{lk}(7)$. If $(a, b) = (3, z)$ then, considering $\text{lk}(3)$ we get $C_4(7, 3, 0, 1) \subseteq \text{lk}(2)$. So, $\text{lk}(w) = C_6(u, 5, 8, 7, z, 3)$.

Completing successively, we get $\text{lk}(3) = C_6(2, 0, 4, z, w, u)$, $\text{lk}(u) = C_6(2, 3, w, 5, 6, v)$, $\text{lk}(2) = C_6(1, 0, 3, u, v, 7)$ and $\text{lk}(z) = C_6(3, 4, 9, v, 7, w)$. Here $M \cong Q_{7,2}$ by the map $\varphi \circ (0, 1, w)(2, z, 4, 3)(5, 9)(6, 7, v)(8, u)$.

Subcase 3.2.3. $\text{lk}(5) = C_6(u, 6, 0, 4, w, z)$. This implies that $\text{lk}(4) = C_6(w, 5, 0, 3, x, y)$, for some $x, y \in V$. It is easy to check that $(x, y) = (7, 2), (7, 8), (7, v), (8, 7), (8, 9), (9, v), (u, v), (v, 7), (v, 8), (v, 9), (v, u), (z, 7), (z, 8), (z, u)$ or (z, v) . By similar arguments as in the previous claims one gets $(x, y) = (8, 7), (8, 9)$ or $(9, v)$.

Subcase 3.2.3.1. $\text{lk}(4) = C_6(w, 5, 0, 3, 8, 9)$. Completing successively, we get $\text{lk}(9) = C_6(w, 4, 8, 1, 6, v)$, $\text{lk}(w) = C_6(v, 9, 4, 5, z, 7)$, $\text{lk}(8) = C_6(3, 4, 9, 1, 7, z)$, $\text{lk}(7) = C_6(2, 1, 8, z, w, v)$, $\text{lk}(z) = C_6(3, 8, 7, w, 5, u)$, $\text{lk}(v) = C_6(2, 7, w, 9, 6, u)$ and $\text{lk}(u) = C_6(2, 3, z, 5, 6, v)$. Here $M \cong T_{14,1,3}$ by the map $\varphi \circ (0, 5, 1, 6, 2, 9, 3, 8, 7)(u, v, w, z)$.

Subcase 3.2.3.2. $\text{lk}(4) = C_6(w, 5, 0, 3, 9, v)$. Completing successively, we get $\text{lk}(9) = C_6(3, 4, v, 6, 1, 8)$, $\text{lk}(3) = C_6(2, 0, 4, 9, 8, z)$, $\text{lk}(v) = C_6(u, 6, 9, 4, w, 7)$, $\text{lk}(2) = C_6(1, 0, 3, z, u, 7)$, $\text{lk}(8) = C_6(1, 9, 3, z, w, 7)$, $\text{lk}(z) = C_6(2, 3, 8, w, 5, u)$, $\text{lk}(w) = C_6(4, 5, z, 8, 7, v)$ and $\text{lk}(7) = C_6(1, 2, u, v, w, 8)$. Here $M \cong T_{14,1,3}$ by the map $\varphi \circ (0, 1, 5, z, w)(3, v, 7, 6, 4, u)(8, 9)$.

Subcase 3.2.3.3. $\text{lk}(4) = C_6(w, 5, 0, 3, 8, 7)$. Completing successively we get $\text{lk}(7) = C_6(w, 4, 8, 1, 2, v)$, $\text{lk}(8) = C_6(3, 4, 7, 1, 9, z)$, $\text{lk}(9) = C_6(z, 8, 1, 6, v, w)$, $\text{lk}(v) = C_6(2, 7, w, 9, 6, u)$, $\text{lk}(z) = C_6(5, w, 9, 8, 3, u)$, $\text{lk}(u) = C_6(2, v, 6, 5, z, 3)$ and $\text{lk}(2) = C_6(1, 0, 3, u, v, 7)$. Here $M \cong Q_{7,2}$ by the map given by $\varphi \circ (0, 1, 3, 7, u, 8, 5, z, 6, 2, 9, 4, w, v)$. \square

LEMMA 3.12. *If M is a 15-vertex degree-regular combinatorial 2-manifolds of Euler characteristic 0 then M is isomorphic to $T_{15,1,2}, \dots, T_{15,1,5}, B_{3,5}, B_{5,3}$ or $Q_{5,3}$.*

Proof. Let M be a 15-vertex degree regular combinatorial 2-manifold of Euler characteristic 0. Let the vertex set V be $\{0, 1, \dots, 9, u, v, w, z, s\}$. Let $\varphi: V \rightarrow \{1, \dots, 15\}$ be given by $\varphi(i) = i$, for $1 \leq i \leq 9$, $\varphi(0) = 10$, $\varphi(u) = 11$, $\varphi(v) = 12$, $\varphi(w) = 13$, $\varphi(z) = 14$ and $\varphi(s) = 15$.

Since $\chi(M) = 0$, the degree of each vertex is 6. As earlier, we may assume that $\text{lk}(0) = C_6(1, 2, 3, 4, 5, 6)$. By Lemma 3.8, $\text{lk}(1) = C_6(7, 2, 0, 6, x, y)$, for some $x, y \in V$. It is easy to see that $(x, y) = (3, 4), (3, 5), (3, 8), (4, 3), (4, 5), (4, 8), (8, 3), (8, 4), (8, 5), (8, 9)$.

If $(x, y) = (3, 5)$ then, considering $\text{lk}(3)$ we get 7 vertices in $\text{lk}(5)$. The case $(x, y) = (8, 3)$ is isomorphic to the case $(x, y) = (4, 8)$ by the map $(0, 1)(2, 6)(3, 4, 8)(5, 7)$ and the case $(x, y) = (8, 5)$ is isomorphic to the case $(x, y) = (8, 3)$ by the map $(2, 6)(3, 5)(7, 8)$. So, we may assume that $(x, y) = (3, 4), (3, 8), (4, 3), (4, 5), (4, 8), (8, 4)$ or $(8, 9)$.

Case 1. $(x, y) = (3, 4)$, i.e., $\text{lk}(1) = C_6(7, 2, 0, 6, 3, 4)$. Then $\text{lk}(3) = C_6(2, 0, 4, 1, 6, 8)$, $\text{lk}(6) = C_6(5, 0, 1, 3, 8, 9)$, $\text{lk}(4) = C_6(5, 0, 3, 1, 7, u)$ and $\text{lk}(2) = C_6(8, 3, 0, 1, 7, v)$. Now, it is easy to see that $\text{lk}(8) = C_6(9, 6, 3, 2, v, u)$ or $C_6(9, 6, 3, 2, v, w)$. In the first case, we get 34 edges in $M[\{0, \dots, 9, u, v\}]$, a contradiction to Lemma 3.9. So, $\text{lk}(8) = C_6(9, 6, 3, 2, v, w)$. Now, completing successively, we get $\text{lk}(7) = C_6(u, 4, 1, 2, v, z)$, $\text{lk}(v) = C_6(w, 8, 2, 7, z, s)$, $\text{lk}(5) = C_6(4, 0, 6, 9, s, u)$, $\text{lk}(s) = C_6(5, 9, z, v, w, u)$, $\text{lk}(u) = C_6(7, 4, 5, s, w, z)$ and $\text{lk}(9) = C_6(5, 6, 8, w, z, s)$. Here $M \cong B_{3,5}$ by the map $\psi_{35} \circ (2, 9, z)(0, v, 6, u, 1, 8)(3, 7, 5, s)$, where $\psi_{35}: V \rightarrow V(B_{3,5})$ given by $\psi_{35}(i) = v_{1i}$, $\psi_{35}(3+i) = v_{2i}$, $\psi_{35}(6+i) = v_{3i}$, $1 \leq i \leq 3$, $\psi_{35}(0) = v_{41}$, $\psi_{35}(u) = v_{42}$, $\psi_{35}(v) = v_{43}$, $\psi_{35}(w) = v_{51}$, $\psi_{35}(z) = v_{52}$ and $\psi_{35}(s) = v_{53}$.

Case 2. $(x, y) = (3, 8)$. Then $\text{lk}(3) = C_6(2, 0, 4, 8, 1, 6)$ or $C_6(2, 0, 4, 6, 1, 8)$.

Subcase 2.1. $\text{lk}(3) = C_6(2, 0, 4, 8, 1, 6)$. Completing successively, we get $\text{lk}(2) = C_6(7, 1, 0, 3, 6, 9)$, $\text{lk}(6) = C_6(5, 0, 1, 3, 2, 9)$, $\text{lk}(9) = C_6(5, 6, 2, 7, v, u)$, $\text{lk}(5) = C_6(4, 0, 6, 9, u, w)$,

$\text{lk}(7) = C_6(8, 1, 2, 9, v, z)$, $\text{lk}(8) = C_6(4, 3, 1, 7, z, s)$, $\text{lk}(4) = C_6(5, 0, 3, 8, s, w)$, $\text{lk}(u) = C_6(5, 9, v, s, z, w)$, $\text{lk}(s) = C_6(4, 8, z, u, v, w)$ and $\text{lk}(z) = C_6(7, 8, s, u, w, v)$. Here M is isomorphic to $Q_{5,3}$ by the map $\psi \circ (5, 8, s, v, 9, 7, 6)(0, 1, 3, 2, 4, z, w, u)$, where $\psi: V \rightarrow V(Q_{5,3})$ is given by $\psi(i) = u_{i1}$, $\psi(5+i) = u_{i2}$, $1 \leq i \leq 3$, $\psi(3+j) = v_{j1}$, $1 \leq j \leq 2$, $\psi(9) = v_{12}$, $\psi(0) = v_{22}$, $\psi(u) = u_{13}$, $\psi(v) = u_{23}$, $\psi(w) = u_{33}$, $\psi(z) = v_{13}$ and $\psi(s) = v_{23}$.

Subcase 2.2. $\text{lk}(3) = C_6(2, 0, 4, 6, 1, 8)$. Completing successively, we get $\text{lk}(2) = C_6(7, 1, 0, 3, 8, 9)$, $\text{lk}(8) = C_6(7, 1, 3, 2, 9, u)$, $\text{lk}(6) = C_6(5, 0, 1, 3, 4, v)$, $\text{lk}(4) = C_6(5, 0, 3, 6, v, w)$, $\text{lk}(7) = C_6(9, 2, 1, 8, u, z)$, $\text{lk}(5) = C_6(v, 6, 0, 4, w, s)$, $\text{lk}(9) = C_6(8, 2, 7, z, s, u)$, $\text{lk}(v) = C_6(6, 4, w, z, s, 5)$, $\text{lk}(s) = C_6(u, w, 5, v, z, 9)$ and $\text{lk}(u) = C_6(7, 8, 9, s, w, z)$. Here $M \cong T_{15,1,2}$ by the map $\varphi \circ (0, 7, 2, 4, 9, 1, 5)(3, 6, 8)(u, s, w, v)$.

Case 3. $(x, y) = (4, 3)$, i.e., $\text{lk}(1) = C_6(7, 2, 0, 6, 4, 3)$. Then $\text{lk}(4) = C_6(5, 0, 3, 1, 6, 8)$, $\text{lk}(3) = C_6(2, 0, 4, 1, 7, 9)$, $\text{lk}(6) = C_6(5, 0, 1, 4, 8, u)$, $\text{lk}(2) = C_6(9, 3, 0, 1, 7, v)$, $\text{lk}(7) = C_6(9, 3, 1, 2, v, w)$, $\text{lk}(5) = C_6(8, 4, 0, 6, u, z)$, $\text{lk}(8) = C_6(u, 6, 4, 5, z, s)$, $\text{lk}(9) = C_6(v, 2, 3, 7, w, s)$. Thus, $\text{lk}(s) = C_6(v, 9, w, z, 8, u)$ or $C_6(v, 9, w, u, 8, z)$. In the first case, considering the links of s, u, v successively, we get 7 vertices in $\text{lk}(v)$. So, $\text{lk}(s) = C_6(v, 9, w, u, 8, z)$. Now, completing successively, we get $\text{lk}(u) = C_6(5, 6, 8, s, w, z)$ and $\text{lk}(z) = C_6(5, 8, s, v, w, u)$. Here $M \cong T_{15,1,2}$ by the map $\varphi \circ (0, 5, 2, 8, 1, 6, 3, 7, 9)(u, s, w, v)$.

Case 4. $(x, y) = (4, 5)$, i.e., $\text{lk}(1) = C_6(7, 2, 0, 6, 4, 5)$. Now, completing successively, we get $\text{lk}(4) = C_6(3, 0, 5, 1, 6, 8)$, $\text{lk}(6) = C_6(5, 0, 1, 4, 8, 9)$, $\text{lk}(5) = C_6(7, 1, 4, 0, 6, 9)$, $\text{lk}(7) = C_6(2, 1, 5, 9, v, u)$, $\text{lk}(2) = C_6(3, 0, 1, 7, u, w)$, $\text{lk}(3) = C_6(8, 4, 0, 2, w, z)$, $\text{lk}(8) = C_6(9, 6, 4, 3, z, s)$, $\text{lk}(9) = C_6(7, 5, 6, 8, s, v)$, $\text{lk}(v) = C_6(u, 7, 9, s, w, z)$, $\text{lk}(z) = C_6(w, 3, 8, s, u, v)$ and $\text{lk}(u) = C_6(2, 7, v, z, s, w)$. Now, $M \cong Q_{5,3}$ by the map $\psi \circ (0, 1, 2, z)(3, 8, 7, s, 9, 6, 4, 5)(u, v, w)$, where ψ is as in Subcase 2.1.

Case 5. $(x, y) = (4, 8)$, i.e., $\text{lk}(1) = C_6(7, 2, 0, 6, 4, 8)$. Now, completing successively, we get $\text{lk}(4) = C_6(5, 0, 3, 6, 1, 8)$, $\text{lk}(6) = C_6(5, 0, 1, 4, 3, 9)$, $\text{lk}(3) = C_6(2, 0, 4, 6, 9, u)$, $\text{lk}(2) = C_6(7, 1, 0, 3, u, v)$, $\text{lk}(5) = C_6(8, 4, 0, 6, 9, w)$, $\text{lk}(8) = C_6(7, 1, 4, 5, w, z)$, $\text{lk}(9) = C_6(u, 3, 6, 5, w, s)$, $\text{lk}(7) = C_6(v, 2, 1, 8, z, s)$, $\text{lk}(s) = C_6(w, 9, u, z, 7, v)$, $\text{lk}(w) = C_6(8, 5, 9, s, v, z)$ and $\text{lk}(z) = C_6(7, 8, w, v, u, s)$. Now, M is isomorphic to $B_{3,5}$ by the map $\psi_{35} \circ (1, u)(2, s)(5, 9)(0, v, 3, 4, 7, z, w, 6, 8)$, where ψ_{35} is as in Case 1.

Case 6. $(x, y) = (8, 4)$, i.e., $\text{lk}(1) = C_6(7, 2, 0, 6, 8, 4)$. Now, $\text{lk}(4) = C_6(5, 0, 3, 8, 1, 7)$ or $C_6(5, 0, 3, 7, 1, 8)$. In the first case, we get $\text{lk}(3) = C_6(8, 4, 0, 2, a, b)$, for some $a, b \in V$. Using Lemma 3.9, we may assume that $(a, b) = (9, u)$. Considering the links of $2, 7, 5, 6, 8, 9, s, u$ and v successively, we get 7 vertices in $\text{lk}(v)$. Thus, $\text{lk}(4) = C_6(5, 0, 3, 7, 1, 8)$.

Again, by using Lemma 3.9, we get $\text{lk}(3) = C_6(7, 4, 0, 2, 9, u)$, $\text{lk}(2) = C_6(7, 1, 0, 3, 9, v)$, $\text{lk}(7) = C_6(2, 1, 4, 3, u, v)$, $\text{lk}(9) = C_6(2, 3, u, z, w, v)$, $\text{lk}(v) = C_6(u, 7, 2, 9, w, s)$, $\text{lk}(u) = C_6(9, 3, 7, v, s, z)$. Then $\text{lk}(z) = C_6(s, u, 9, w, a, b)$, for some $a, b \in V$. It is easy to see that $(a, b) = (5, 6), (5, 8), (6, 5), (6, 8), (8, 5)$ or $(8, 6)$. Since the set of known faces remain invariant under the map $(0, 4)(2, 7)(6, 8)(9, u)(w, s)$, we can assume that $(a, b) = (5, 6), (5, 8), (6, 8)$ or $(8, 6)$.

Subcase 6.1. $(a, b) = (5, 6)$. Completing successively, we get $\text{lk}(5) = C_6(8, 4, 0, 6, z, w)$, $\text{lk}(6) = C_6(8, 1, 0, 5, z, s)$ and $\text{lk}(8) = C_6(5, 4, 1, 6, s, w)$. Now, M is isomorphic to $B_{5,3}$ by the map $\psi_{53} \circ (0, s, 3, 1)(2, 6, 9, 7, u)(4, 5, z, 8)$, where $\psi_{53}: V \rightarrow V(B_{5,3})$ is given by $\psi_{53}(i) = v_{1i}$, for $1 \leq i \leq 5$, $\psi_{53}(5+i) = v_{2i}$, for $1 \leq i \leq 4$, $\psi_{53}(0) = v_{25}$, $\psi_{53}(u) = v_{31}$, $\psi_{53}(v) = v_{32}$, $\psi_{53}(w) = v_{33}$, $\psi_{53}(z) = v_{34}$, $\psi_{53}(s) = v_{35}$.

Subcase 6.2. $(a, b) = (5, 8)$. Completing successively, we get $\text{lk}(5) = C_6(w, z, 8, 4, 0, 6)$, $\text{lk}(8) = C_6(6, 1, 4, 5, z, s)$ and $\text{lk}(6) = C_6(5, 0, 1, 8, s, w)$. Here M is isomorphic to $T_{15,1,5}$ by the map $\varphi \circ (0, 2, 1, 7, 6, 8, w, 9)(3, u, 5)(4, v, s, z)$.

Subcase 6.3. $(a, b) = (6, 8)$. Completing successively, we get $\text{lk}(6) = C_6(5, 0, 1, 8, z, w)$, $\text{lk}(8) = C_6(1, 4, 5, s, z, 6)$ and $\text{lk}(5) = 4, 0, 6, w, s, 8$. Here $M \cong B_{5,3}$ by the map $\psi_{53} \circ (1, 9)(0, z, 2, s, v, 6, 8, 3, 5, w, 7)$, where ψ_{53} is as in Subcase 6.1.

Subcase 6.4. $(a, b) = (8, 6)$. Completing successively, we get $\text{lk}(6) = C_6(5, 0, 1, 8, z, s)$, $\text{lk}(8) = C_6(5, 4, 1, 6, z, w)$ and $\text{lk}(5) = C_6(8, 4, 0, 6, s, w)$. Here M is isomorphic to $T_{15,1,4}$ by the map given by $\varphi \circ (0, 7, 1, 2, 6, 3, u, s, 4, v, 5, 8, w, 9)$.

Case 7. $(x, y) = (8, 9)$, i.e., $\text{lk}(1) = C_6(7, 2, 0, 6, 8, 9)$. Now, $\text{lk}(6) = C_6(8, 1, 0, 5, a, b)$. It is easy to check that $(a, b) = (2, 3), (2, 7), (3, 2), (3, 7), (3, 4), (3, u), (7, 2), (7, 3), (7, 4), (7, u), (9, 3), (9, 4), (9, 7), (9, u), (u, 3), (u, 4), (u, 7), (u, v)$.

If $(a, b) = (3, 7)$ then, considering links of 3 and 7, we get 7 vertices in $\text{lk}(7)$. If $(a, b) = (7, 3), (7, 4)$ or $(9, 4)$ then, considering $\text{lk}(b)$, we get 7 vertices in $\text{lk}(a)$. Since the set of known faces remain invariant under that map $(0, 1)(3, 7)(4, 9)(5, 8)$, we may assume that $(a, b) = (2, 3), (2, 7), (3, 4), (3, u), (7, u), (9, u), (u, v)$.

Subcase 7.1. $\text{lk}(6) = C_6(8, 1, 0, 5, 2, 3)$. Completing successively, we get $\text{lk}(2) = C_6(1, 0, 3, 6, 5, 7)$, $\text{lk}(5) = C_6(4, 0, 6, 2, 7, u)$, $\text{lk}(3) = C_6(4, 0, 2, 6, 8, v)$, $\text{lk}(8) = C_6(9, 1, 6, 3, v, w)$, $\text{lk}(4) = C_6(u, 5, 0, 3, v, z)$, $\text{lk}(v) = C_6(w, 8, 3, 4, z, s)$, $\text{lk}(7) = C_6(9, 1, 2, 5, u, s)$, $\text{lk}(9) = C_6(w, 8, 1, 7, s, z)$, $\text{lk}(z) = C_6(4, v, s, 9, w, u)$ and $\text{lk}(s) = C_6(w, u, 7, 9, z, v)$. Here $M \cong B_{3,5}$ by the map $\psi_{35} \circ (0, 7, 5, 9, 1, 4)(2, 8, s)(3, u, 6, v, z, w)$, where ψ_{35} is as in Case 1.

Subcase 7.2. $\text{lk}(6) = C_6(8, 1, 0, 5, 2, 7)$. Completing successively, we get $\text{lk}(2) = C_6(5, 6, 7, 1, 0, 3)$, $\text{lk}(5) = C_6(4, 0, 6, 2, 3, u)$, $\text{lk}(3) = C_6(4, 0, 2, 5, u, v)$, $\text{lk}(4) = C_6(u, 5, 0, 3, v, w)$, $\text{lk}(u) = C_6(v, 3, 5, 4, w, z)$, $\text{lk}(7) = C_6(9, 1, 2, 6, 8, s)$, $\text{lk}(v) = C_6(w, 4, 3, u, z, s)$, $\text{lk}(8) = C_6(9, 1, 6, 7, s, z)$, $\text{lk}(9) = C_6(8, 1, 7, s, w, z)$ and $\text{lk}(s) = C_6(8, 7, 9, w, v, z)$. Here $M \cong T_{15,1,2}$ by the map $\varphi \circ (0, 7, 3, 9, 1, 4)(2, 6, 5, 8)$.

Subcase 7.3. $\text{lk}(6) = C_6(8, 1, 0, 5, 3, 4)$. Completing successively, we get $\text{lk}(3) = C_6(2, 0, 4, 6, 5, u)$, $\text{lk}(4) = C_6(5, 0, 3, 6, 8, v)$, $\text{lk}(5) = C_6(3, 6, 0, 4, v, u)$, $\text{lk}(2) = C_6(7, 1, 0, 3, u, w)$, $\text{lk}(u) = C_6(v, 5, 3, 2, w, z)$, $\text{lk}(v) = C_6(8, 4, 5, u, z, s)$, $\text{lk}(8) = C_6(9, 1, 6, 4, v, s)$, $\text{lk}(9) = C_6(7, 1, 8, s, w, z)$, $\text{lk}(w) = C_6(2, 7, s, 9, z, u)$ and $\text{lk}(7) = C_6(9, 1, 2, w, s, z)$. Here $M \cong Q_{5,3}$ by the map $\psi \circ (0, 1, 8, 7, u, s, 9)(2, z, w, v, 6, 5, 3)$, where ψ is as in Subcase 2.1.

Subcase 7.4. $\text{lk}(6) = C_6(8, 1, 0, 5, 3, u)$. Completing successively, we get $\text{lk}(3) = C_6(4, 0, 2, 5, 6, u)$, $\text{lk}(2) = C_6(7, 1, 0, 3, 5, v)$, $\text{lk}(5) = C_6(4, 0, 6, 3, 2, v)$, $\text{lk}(4) = C_6(u, 3, 0, 5, v, w)$, $\text{lk}(u) = C_6(8, 6, 3, 4, w, z)$, $\text{lk}(8) = C_6(9, 1, 6, u, z, s)$, $\text{lk}(v) = C_6(7, 2, 5, 4, w, s)$, $\text{lk}(7) = C_6(9, 1, 2, v, s, z)$, $\text{lk}(s) = C_6(9, w, v, 7, z, 8)$ and $\text{lk}(z) = C_6(w, 9, 7, s, 8, u)$. Here $M \cong B_{3,5}$ by the map $\psi_{35} \circ (0, 7, z, 2, u, 5, v, s, 3, 8, 6, 9, w, 1)$, where ψ_{35} is as defined in Case 1.

Subcase 7.5. $\text{lk}(6) = C_6(8, 1, 0, 5, 7, u)$. Now, it is easy to see that $\text{lk}(7) = C_6(9, 1, 2, 5, 6, u)$ or $C_6(9, 1, 2, u, 6, 5)$. The first case is isomorphic to Subcase 6.2 by that map $(0, 3, 9, 6, 4, u, 8, 5, 7, 1)$. The second case is isomorphic to Subcase 6.1 by the map $(0, 3, 9, 6, 4, u, 7, 1)(5, 8)$.

Subcase 7.6. $\text{lk}(6) = C_6(8, 1, 0, 5, 9, u)$. Completing successively, we get $\text{lk}(9) = C_6(8, 1, 7, u, 6, 5)$, $\text{lk}(5) = C_6(4, 0, 6, 9, 8, v)$, $\text{lk}(8) = C_6(6, 1, 9, 5, v, u)$, $\text{lk}(u) = C_6(7, 9, 6, 8, v, w)$, $\text{lk}(7) = C_6(2, 1, 9, u, w, z)$, $\text{lk}(v) = C_6(4, 5, 8, u, w, s)$, $\text{lk}(2) = C_6(3, 0, 1, 7, z, s)$, $\text{lk}(s) = C_6(3, 2, z, 4, v, w)$, $\text{lk}(4) = C_6(3, 0, 5, v, s, z)$ and $\text{lk}(3) = C_6(4, 0, 2, s, w, z)$. Here $M \cong B_{3,5}$ by the map $\psi_{35} \circ (1, z, v, 5, 6, w, 4, 9, 3, 7, s, 8, 2, u)$, where ψ_{35} is as in Case 1.

Subcase 7.7. $\text{lk}(6) = C_6(8, 1, 0, 5, u, v)$. Then, $\text{lk}(5) = C_6(u, 6, 0, 4, c, d)$, for some $c, d \in V$. It is easy to see that $(c, d) = (2, 3), (2, 7), (7, 2), (7, 3), (7, 9), (7, w), (8, 9), (9, 3), (9, 7), (9, 8), (9, w), (v, 3), (v, 7), (v, 8), (v, 9), (v, w), (w, 3), (w, 7), (w, 9), (w, z)$. Since the set of known faces remain invariant under the map $(0, 6)(2, 8)(3, v)(4, u)(7, 9)$, we may assume that $(c, d) = (2, 3), (2, 7), (7, 2), (7, 3), (7, 9), (7, w), (9, 3), (9, 7), (9, w), (v, 3), (v, w), (w, z)$.

Claim. $(c, d) = (2, 3), (9, w), (v, w)$ or (w, z) .

If $(c, d) = (2, 7)$ then, considering $\text{lk}(2)$ we get $C_4(3, 2, 5, 0) \subseteq \text{lk}(4)$. If $(c, d) = (7, 3)$ then, considering $\text{lk}(3)$, we get $C_4(7, 3, 0, 1) \subseteq \text{lk}(2)$.

If $(c, d) = (7, 2)$ then, considering the links of $2, u, 3, 4, 7, w, 9, v$ successively, we get 7 vertices in $\text{lk}(8)$. If $(c, d) = (9, 3)$ or $(v, 3)$ then, considering $\text{lk}(3)$, we get 7 vertices in $\text{lk}(c)$. If $(c, d) = (9, 7)$ then, considering the links of $9, 4, 7, 2, v, 8, s$ successively, we get 7 vertices in $\text{lk}(u)$. If $(c, d) = (7, 9)$ then, considering the links of $5, 7, 2, 4, 3, v$ successively, we get $\text{lk}(u) = C_6(9, 5, 6, v, z, x)$, where $x = z$ or s . In either case $\text{lk}(x)$ has ≥ 7 vertices.

If $(c, d) = (7, w)$ then, $\text{lk}(7) = C_6(2, 1, 9, w, 5, 4)$ or $C_6(2, 1, 9, 4, 5, w)$. In the first case, considering the links of 4 and 2 we get $C_4(z, 2, 0, 4) \subseteq \text{lk}(3)$. In the second case, considering links of $7, 4, 2, w, z, u$ successively, we get 7 vertices in $\text{lk}(3)$. This proves the claim.

Subcase 7.7.1. $\text{lk}(5) = C_6(u, 6, 0, 4, 2, 3)$. Now, completing successively we get $\text{lk}(2) = C_6(1, 0, 3, 5, 4, 7)$, $\text{lk}(4) = C_6(3, 0, 5, 2, 7, w)$, $\text{lk}(3) = C_6(5, 2, 0, 4, w, u)$, $\text{lk}(u) = C_6(6, 5, 3, w, s, v)$, $\text{lk}(7) = C_6(9, 1, 2, 4, w, z)$, $\text{lk}(w) = C_6(3, 4, 7, z, s, u)$, $\text{lk}(8) = C_6(9, 1, 6, v, z, s)$, $\text{lk}(9) = C_6(8, 1, 7, z, v, s)$ and $\text{lk}(v) = C_6(8, 6, u, s, 9, z)$. Here M is isomorphic to $Q_{5,3}$ by the map $\psi \circ (0, 1, z, w, 6, 8, u, 7, s, 9, v)(3, 4)$, where ψ is as in Subcase 2.1.

Subcase 7.7.2. $\text{lk}(5) = C_6(u, 6, 0, 4, 9, w)$. This implies that $\text{lk}(9) = C_6(7, 1, 8, 4, 5, w)$ or $C_6(7, 1, 8, w, 5, 4)$. In the first case, considering links of $9, 4, 8, v$ successively we see that $\text{lk}(v)$ can not be a 6-cycle. Thus $\text{lk}(9) = C_6(7, 1, 8, w, 5, 4)$.

Now, completing successively, we get $\text{lk}(4) = C_6(3, 0, 5, 9, 7, z)$, $\text{lk}(7) = C_6(2, 1, 9, 4, z, s)$, $\text{lk}(2) = C_6(3, 0, 1, 7, s, v)$, $\text{lk}(3) = C_6(z, 4, 0, 2, v, u)$, $\text{lk}(u) = C_6(5, 6, v, 3, z, w)$, $\text{lk}(v) = C_6(6, u, 3, 2, s, 8)$, $\text{lk}(8) = C_6(9, 1, 6, v, s, w)$ and $\text{lk}(z) = C_6(7, 4, 3, u, w, s)$. Here $M \cong T_{15,1,3}$ by the map $\varphi \circ (0, 5, 8)(1, 6, 9, 7, 3)(u, v, w)(z, s)$.

Subcase 7.7.3. $\text{lk}(5) = C_6(u, 6, 0, 4, v, w)$. Now, completing successively we get $\text{lk}(v) = C_6(8, 6, u, 4, 5, w)$, $\text{lk}(4) = C_6(3, 0, 5, v, u, z)$, $\text{lk}(u) = C_6(5, 6, v, 4, z, w)$, $\text{lk}(w) = C_6(8, v, 5, u, z, s)$, $\text{lk}(8) = C_6(9, 1, 6, v, w, s)$, $\text{lk}(z) = C_6(3, 4, u, w, s, 7)$, $\text{lk}(7) = C_6(2, 1, 9, 3, z, s)$, $\text{lk}(3) = C_6(2, 0, 4, z, 7, 9)$ and $\text{lk}(2) = C_6(7, 1, 0, 3, 9, s)$. Here M is isomorphic to $B_{3,5}$ by the map $\psi_{35} \circ (1, 9, 8, 5, w)(2, 7, v)(3, u)(4, z, s)$, where ψ_{35} is as in Case 1.

Subcase 7.7.4. $\text{lk}(5) = C_6(u, 6, 0, 4, w, z)$. Then $\text{lk}(4) = C_6(w, 5, 0, 3, x, y)$, for some $x, y \in V$. It is easy to see that $(x, y) = (7, 2), (7, 9), (7, v), (7, s), (8, 9), (8, v), (9, 8), (9, v), (9, s), (u, v), (v, 8), (v, 9), (v, u), (v, s), (z, 9), (z, u), (z, v), (z, s), (s, 9), (s, v)$. By the similar arguments as before one gets $(x, y) = (7, 2), (9, 8)$ or (z, u) .

Subcase 7.7.4.1. $\text{lk}(4) = C_6(w, 5, 0, 3, 9, 8)$. Completing successively, we get $\text{lk}(8) = C_6(6, 1, 9, 4, w, v)$, $\text{lk}(9) = C_6(7, 1, 8, 4, 3, s)$, $\text{lk}(3) = C_6(2, 0, 4, 9, s, z)$, $\text{lk}(2) = C_6(7, 1, 0, 3, z, u)$, $\text{lk}(u) = C_6(5, 6, v, 7, 2, z)$, $\text{lk}(7) = C_6(9, 1, 2, u, v, s)$, $\text{lk}(v) = C_6(8, 6, u, 7, s, w)$ and $\text{lk}(w) = C_6(5, 4, 8, v, s, z)$. Here $M \cong T_{15,1,3}$ by the map $\varphi \circ (0, 6, 9, 4, 7, 1, 5)(u, w)$.

Subcase 7.7.4.2. $\text{lk}(4) = C_6(w, 5, 0, 3, z, u)$. Completing successively, we get $\text{lk}(u) = C_6(6, 5, z, 4, w, v)$, $\text{lk}(z) = C_6(w, 5, u, 4, 3, s)$, $\text{lk}(w) = C_6(u, 4, 5, z, s, v)$, $\text{lk}(3) = C_6(2, 0, 4, z, s, 9)$, $\text{lk}(9) = C_6(3, 2, 8, 1, 7, s)$, $\text{lk}(s) = C_6(w, z, 3, 9, 7, v)$, $\text{lk}(v) = C_6(6, u, w, s, 7, 8)$ and $\text{lk}(7) = C_6(2, 1, 9, s, v, 8)$. Here $M \cong Q_{5,3}$ by the map $\psi \circ (0, z, w, 9, 3, s, 6, 8, 5, u)(4, v, 7)$, where ψ is as in Subcase 2.1.

Subcase 7.7.4.3. $\text{lk}(4) = C_6(w, 5, 0, 3, 7, 2)$. Then, completing successively, we get $\text{lk}(2) = C_6(3, 0, 1, 7, 4, w)$, $\text{lk}(3) = C_6(7, 4, 0, 2, w, s)$, $\text{lk}(7) = C_6(1, 2, 4, 3, s, 9)$, $\text{lk}(w) = C_6(5, 4, 2, 3, s, z)$, $\text{lk}(s) = C_6(9, 7, 3, w, z, v)$, $\text{lk}(v) = C_6(u, 6, 8, z, s, 9)$, $\text{lk}(z) = C_6(5, w, s, v, 8, u)$ and $\text{lk}(8) = C_6(9, 1, 6, v, z, u)$. Here $M \cong B_{3,5}$ by the map $\psi_{35} \circ (0, 9, 1, 5)(2, 8)(3, v)(4, 7)(u, w)$, where ψ_{35} is as in Case 1. \square

Proof of Theorem 6. Let M be an n -vertex degree-regular combinatorial 2-manifold of Euler characteristic 0. Let d be the degree of each vertex. Then $nd = 2f_1(M) = 3f_2(M)$ and $n - f_1(M) + f_2(M) = 0$. These imply that $d = 6$. Now, if $n \in \{12, 14, 15\}$ then, by Lemmas 3.10, ..., 3.12, M is isomorphic to $T_{12,1,2}, \dots, T_{12,1,4}, T_{6,2,2}, T_{14,1,2}, T_{14,1,3}, Q_{7,2}, T_{15,1,2}, \dots, T_{15,1,5}, Q_{5,3}, B_{3,4}, B_{4,3}, B_{3,5}, B_{5,3}$ or $K_{3,4}$.

Since $B_{3,4}, B_{4,3}, B_{3,5}, B_{5,3}, Q_{7,2}, Q_{5,3}$ and $K_{3,4}$ are non-orientable and remaining 10 are orientable, the second and third statements follow from Lemma 2.1 (b), (d), (f), Lemma 2.3 (b) and Lemma 2.5 (a), (b).

The last statement follows from the fact that $B_{m,n}, K_{m,2k}, Q_{2k+1,n}$ are not weakly regular and $Q_{2k+1,2}$ is weakly regular for all $m, n \geq 3$ and $k \geq 2$. \square

Acknowledgement : The authors are thankful to B. Bagchi, S. P. Inamdar and N. S. N. Sastry for useful conversations. The authors thank the anonymous referee for many useful comments which led to substantial improvements in the presentation of this paper. Theorem 1 was conjectured by the referee. The second author thanks CSIR, New Delhi, India for its research fellowship (Award No.: 9/79(797)/2001 - EMR - I).

References

- [1] A. Altshuler, Polyhedral realization in \mathbb{R}^3 of triangulations of the torus and 2-manifolds in cyclic 4-polytopes, *Discrete Math.* **1** (1971), 211–238.
- [2] A. Altshuler, Construction and representation of neighbourly manifolds, *J. Comb. Th. (A)* **77** (1997), 246–267.
- [3] A. Altshuler, J. Bokowski and P. Schuchert, Neighbourly 2-manifolds with 12 vertices, *J. Comb. Th. (A)* **75** (1996), 148–162.
- [4] B. Bagchi and B. Datta, A structure theorem for pseudomanifolds, *Discrete Math.* **168** (1998), 41–60.
- [5] B. Datta and N. Nilakantan, Equivelar polyhedra with few vertices, *Discrete & Comput Geom.* **26** (2001), 429–461.
- [6] B. Datta and A. K. Upadhyay, Degree-regular triangulations of the double-torus (preprint).
- [7] M. Jungerman and G. Ringel, Minimal triangulations on orientable surfaces, *Acta Math.* **145** (1980), 121–154.
- [8] F. H. Lutz, *Triangulated Manifolds with Few Vertices and Vertex-Transitive Group Actions*, Thesis (TU, Berlin), Shaker Verlag, Aachen, 1999.
- [9] G. Ringel, Wie man die geschlossenen nichtorientierbaren Flächen in möglichst wenig Dreiecke zerlegen kann, *Math. Ann.* **130** (1955), 317–326.
- [10] E. Schulte and J. M. Wills, A polyhedral realization of Felix Klein’s map $\{3, 7\}_8$ on a Riemann surface of genus 3, *J. London Math. Soc.* **32** (1985), 539–547.
- [11] E. Schulte and J. M. Wills, Geometric realizations for Dyck’s regular map on a surface of genus 3, *Discrete & Comput Geom.* **1** (1986), 141–153.
- [12] E. H. Spanier, *Algebraic Topology*, Springer-Verlag, New York, 1966.